An approach to characterizing the local Langlands conjecture over p-adic fields

Alex Youcis (IMPAN)
(joint with A. Bertoloni Meli (Michigan))

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Some notation

Throughout this talk we will use the following notation:

- F a p-adic local field (with ring of integers $\mathcal{O}$ and residue field k).
- G a reductive group over F (e.g. $G = \mathrm{Sp}_{2n}$).
- $\widehat{G}$ will denote the dual group to G (e.g. $\widehat{\mathrm{Sp}_{2n}} = \mathrm{SO}_{2n+1}$).
- ${}^L G$ will denote the Weil form of the L-group (i.e. ${}^L G = \widehat{G} \rtimes W_F$).

The local Langlands conjecture

A Langlands correspondence

A *Langlands correspondence* for G is a finite-to-one association

$$\mathrm{LL} : \left\{\begin{matrix}\text{Admissible representations}\\ \text{of } G(F)\end{matrix}\right\} \to \left\{\begin{matrix}L\text{-parameters}\\ \text{of } G\end{matrix}\right\}$$

satisfying some list of desiderata.

L-packets

An *L-packet* for LL is a set (necessarily finite) of the form $\Pi_G(\psi) := \mathrm{LL}^{-1}(\psi)$ for some L-parameter ψ of G.

The local Langlands conjecture (cont.)

Some known cases

- $G = \mathrm{GL}_n$ (Harris–Taylor, Henniart, Scholze).
- $G = \mathrm{Sp}_{2n}$ and $G = \mathrm{SO}_{2n+1}$ (Arthur).
- G is a quasi-split unitary group (Mok).
- G is an arbitrary unitary group (Kaletha–Minguez–Shin–White).
- $G = \mathrm{GSp}_4$ (Gan–Takeda)

The local Langlands conjecture (cont.)

Question

What are the desiderata that LL should satisfy and, do they uniquely characterize the correspondence LL?

The GL_n case

In the case of $G = \mathrm{GL}_n$ one can take the standard desiderata that LL is compatible with tensor products, local class field theory, L-functions, and ϵ-factors. These properties do uniquely characterize the correspondence.

Scholze's characterization of LLC for GL_n

Theorem (Scholze, 2013)

For every $\tau \in W_F^+$ and $h \in C_c^\infty(\mathrm{GL}_n(\mathcal{O}), \mathbb{Q})$ there exists a function $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F), \mathbb{Q})$ such that for any admissible representation π of $\mathrm{GL}_n(F)$ the equality

$$\mathrm{tr}(f_{\tau,h} \mid \pi) = \mathrm{tr}(\tau \mid \mathsf{LL}(\pi)(\chi))\,\mathrm{tr}(h \mid \pi)$$

holds, and this uniquely characterizes LL.

The function $f_{\tau,h}$

The functions $f_{\tau,h}$

The functions $f_{\tau,h}$, which really should be denoted $f^{\mu}_{\tau,h}$, are of a geometric provenance.

- They can be described in terms of the cohomology of tubular neighborhoods inside of Rapoport–Zink spaces.
- They show up as terms in the trace formula existing within the framework of the Langlands–Kottwitz(–Scholze) method.
- They have generalized versions beyond GL_n for more general PEL type situations (Scholze) and in most abelian type situations (forthcoming work of the author).

The Scholze–Shin conjecture

Conjecture (Scholze-Shin)

Let G be an unramified group over $\mathbb{Q}_p$ with $\mathbb{Z}_p$-model $\mathcal{G}$ and let μ be a dominant cocharacter of $G_{\overline{\mathbb{Q}_p}}$ with reflex field E. Let $\tau \in W^+_{\mathbb{Q}_p}$ and let $h \in C_c^\infty(\mathcal{G}(\mathbb{Z}_p), \mathbb{Q})$. Then, for every supercuspidal L-parameter ψ

$$S\Theta_\psi(f_{\tau,h}) = \mathrm{tr}\left(\tau \mid (r_{-\mu} \circ \psi) \mid_{W_E} |\cdot|_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\psi(h).$$

Notation

- $S\Theta_\psi(f) := \sum_{\pi \in \Pi_G(\psi)} r_\pi \, \mathrm{tr}(f \mid \pi)$—the *stable character for* ψ.
- $r_{-\mu}$ is the representation of ${}^L G$ whose restriction to $\widehat{G}$ has highest weight $\mu^\vee$.

The Scholze–Shin conjecture (cont.)

First natural question

Does the Scholze–Shin conjecture hold for LL for groups other than GL_n?

Second natural question

Does the Scholze–Shin equations uniquely characterize LL for groups other than GL_n?

The Scholze–Shin conjecture (cont.)

Towards the first question

This is known to hold in some cases:

- Some PEL cases (Scholze and Scholze–Shin).
- It's known to hold (appropriately interpreted) in some cases of the form $G = D^\times$ (Shen).
- It's known to hold in the case of unitary groups (Bertoloni Meli–Y.)

Setup for main result

Supercuspidal L-parameters

An L-parameter ψ is called *supercuspidal* if $\mathrm{im}(\psi)$ does not lie in a proper parabolic subgroup of ${}^LG$ and $\psi\,|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial.

Slogan for supercuspidal L-parameters

Supercuspidal L-parameters should be those whose L-packet consists entirely of supercuspidals.

Setup for main result (cont.)

Elliptic hyperendoscopic group

An *extended elliptic hyperendoscopic datum* is a sequence of tuples of data $(H_1, s_1, {}^L\eta_1), \ldots, (H_k, s_k, {}^L\eta_k)$ such that $(H_1, s_1, {}^L\eta_1)$ is an extended elliptic endoscopic datum of G, and for $i > 1$, the tuple $(H_i, s_i, {}^L\eta_i)$ is an extended elliptic endoscopic datum of H_{i-1}.

Slogan for hyperendoscopic groups

They are a set of groups for which an L-parameter could factorize through and which is sufficiently large to study the packet structure of a parameter.

Example for hyperendoscopic groups

If $G = U(n)$ then the elliptic hyperendoscopic groups for G are those of the form $U(a_1) \times \cdots \times U(a_m)$ with $a_1 + \cdots + a_m = n$.

Setup for main result (cont.)

Supercuspidal local Langlands correspondence

A *supercuspidal local Langlands correspondence* for a group G (assumed quasi-split for simplicity) is an association

$$\Pi_H : \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{Supercuspidal } L\text{-parameters} \\ \text{for } H \end{array} \right\} \to \left\{ \begin{array}{c} \text{Finite subsets} \\ \text{of } \mathrm{Irr}^{\mathrm{sc}}(H(F)) \end{array} \right\}$$

for every elliptic hyperendoscopic group H of G satisfying the following conditions:

- **(Dis)** Packets for distinct parameters are disjoint.
- **(Bij)** There is a bijection between $\Pi_H(\psi)$ and $\mathrm{Irr}(\overline{C_\psi})$.
- **(St)** The 'stable character' $S\Theta_\psi$ for any ψ is actually stable.
- **(ECI)** The endoscopic character identities hold.

Setup for main result (cont.)

Scholze–Shin data

For a group G an *Scholze–Shin datum* is a collection of functions $\varphi^{\mu}_{\tau,h}$ depending on:

- (a certain set) of dominant cocharacters μ of hyperendoscopic groups H.
- $\tau \in W_E^+$ (E the reflex field of μ).
- $h \in C_c^\infty(K_H, \mathbb{Q})$ (K_H some compact open subgroup of $H(F)$).

Scholze–Shin equations

We say that Π_H satisfies the *Scholze–Shin equations* relative to $\{\varphi^{\mu}_{\tau,h}\}$ if the following equation holds for all ψ:

$$S\Theta_\psi(\varphi_{\tau,h}) = \mathrm{tr}\left(\tau \mid (r_{-\mu} \circ \psi)\,|_{W_E} \mid \cdot\,|_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\psi(h).$$

The main result

Theorem (Bertoloni Meli–Y.)

Suppose that G is a 'good' group and that Π^1 and Π^2 are two supercuspidal Langlands correspondences for G which satisfy the Scholze–Shin equations for the same Scholze–Shin datum $\{\varphi^{\mu}_{\tau,h}\}$. Then, $\Pi^1 = \Pi^2$ and the bijections in **(Bij)** *are the same.*

The notion of 'good'

Good groups

A group G is *good* if an L-parameter ψ for any hyperendoscopic groups H is determined by the set of Galois representations $\{r_{-\mu} \circ \psi\}$ for all cocharacters μ.

Examples/Non-examples

Some examples of 'good groups':

Examples	Non-examples
GL_n	SL_n
U_n	SO_{2n}
SO_{2n+1}	Sp_{2n}
PGL_n	E_8
G_2	

Broad idea of proof

- If $\Pi^1_H(\psi) = \{\pi\} = \Pi^2_H(\psi')$ then taking h such that $\Theta_\pi(h) \neq 0$ we see that
$$\mathrm{tr}(\tau \mid r_{-\mu} \circ \psi) = \frac{\Theta_\psi(\varphi^\mu_{\tau,h})}{\Theta_\pi(h)} = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi')$$
implies that $\psi \sim \psi'$.
- The desiderata for a supercuspidal Langlands correspondence non-trivially satisfy *atomic stability*—the fact that if S is a set of representations for which some linear combination is stable, then S is a union of L-packets—this in turn implies that if $\Pi^1_H(\psi) = \{\pi\}$ then $\{\pi\} = \Pi^2_H(\psi')$ for some ψ'.
- If $\Pi^1_H(\psi) = \{\pi\}$ then $\Pi^1_H(\psi) = \Pi^2_H(\psi)$.
- Every ψ can be written as $\eta \circ \psi^{H'}$ for $\psi^{H'}$ a parameter of some hyperendoscopic group H' of H (and thus of G) such that $\Pi^1_{H'}(\psi^{H'})$ is a singleton.

An application

Theorem (Bertoloni Meli–Y.)

Let $E/\mathbb{Q}_p$ be an unramified extension and F the quadratic subextension of E. Let G be the quasi-split unitary group $U_{E/F}(n)^*$ associated to E/F. Then, the local Langlands correspondence for G (as in the work of Mok) satisfies the Scholze–Shin conjecture and is uniquely characterized by this condition.

Further directions and related ideas

- Result is reminiscient of a 'stable version' of V. Lafforgue's lemma following the work of Richardson.
- This 'stable version' is useful when applying global methods since usually only stable characters are accessible.
- Opens up question of whether there is a useful version of our main theorem with Scholze–Shin datum replaced by more general datum indexed by triples $(I, f, \{\gamma_i\})$ as in Lafforgue's lemma.
- Opens up question of whether there is a useful version of our result in the function field setting.

Thanks for listening!