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1 Our first proof

Theorem: $\sqrt{2}$ is irrational.

Definition: A number x is
- **rational** if $x = \frac{a}{b}$ w/ a, b integers,
- **irrational** otherwise.

Theorem: assertion of mathematical truth.

An **integer** is an element of $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

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Text: Proof and the Art of Mathematics by Joel David Hamkins

Evaluation:

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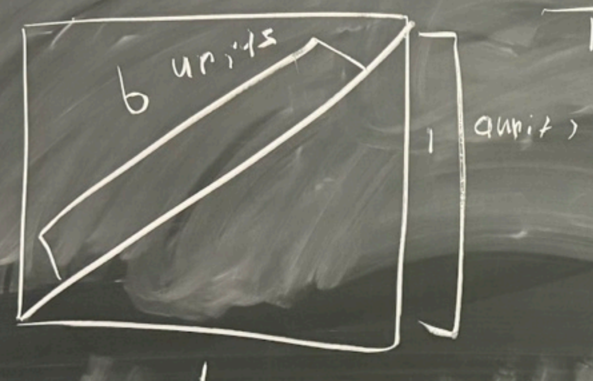
Practice Midterm/Final	7.5+7.5
Quizzes (5 q.)	25
Midterm	25
Final	35

3 Geometric meaning:



a	b
10	14
100	141

$\sqrt{2} = 1.41\ldots$

$\boxed{4}$ Pf: Suppose $\sqrt{2} = \frac{a}{b}$ w/ $a, b \in \mathbb{Z}$

By cancelling common factors of 2 from a and b, we may assume that not both are even. Assume this is the case.

If $\sqrt{2} = \frac{a}{b} \implies 2b^2 = a^2$. But, this implies that a^2 is even. This implies a is even. So, there is $k \in \mathbb{Z}$ s.t. $a = 2k$.

So, $2b^2 = a^2 = (2k)^2 = 4k^2$. So, $b^2 = 2k^2$.

The same logic shows that b is even. Contradiction $\boxtimes$

$\in$: is an element

$\implies$ the / implies

s.t. = such that

Proof by contradiction: Assume the opp. and arrive at absurdity.

2 Our first lemmas

Lemma 1: If $x \in \mathbb{Z}$ is odd, then x^2 is odd.

Pf: If x is odd, then $x = 2k+1$ for $k \in \mathbb{Z}$. Then

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

But, $2k^2 + 2k \in \mathbb{Z}$. Thus, $x^2 = 2l + 1$ for $l = 2k^2 + 2k \in \mathbb{Z}$. Thus, x^2 is odd $\blacksquare$

Lemma: Statement of mathematical truth

"Direct proof" — not a proof by contradiction.

$\blacksquare$ = end of proof

QED

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Lemma 2: Every $\frac{a}{b}$ w/ $a, b \in \mathbb{Z}$ can be written as $\frac{a'}{b'}$ where a', b' are coprime.

e.g.

- 2 and 6 are not coprime
- 1 and 3 are coprime

$\frac{2}{6} = \frac{1}{3}$.

Coprime: $x, y \in \mathbb{Z}$ are coprime if their only common divisors are ± 1.

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Axiom (Least number principle, LNP):

If S is a collection of natural numbers (not empty) then S has a smallest element.

Natural numbers:

element of

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Axiom: Assumed fact

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Pf of Lemma 2: Set $S = \{x \in \mathbb{N} : \frac{a}{b} = \frac{x}{y}$ for some $y \in \mathbb{Z}\}$

First, observe that S is non-empty. Indeed $\frac{a}{b} = \frac{-a}{-b}$ and either a or $-a$ is in $\mathbb{N}$. So, either $a \in S$ or $-a \in S$. Either way, S is non-empty. By LNP there is a smallest

$\{x \in S : \_\_\_\_\}$

$= \{$All elements x of S satisfying $\_\_\_\_\}$

element x_0 of S. By definition, there is some $y_0 \in \mathbb{Z}$ s.t. $\frac{a}{b} = \frac{x_0}{y_0}$.

<u>Claim:</u> x_0 and y_0 are coprime.

<u>Pf:</u> Assume not, then there is some $\pm 1 \neq d \in \mathbb{Z}$ s.t. $d \mid x_0$ and $d \mid y_0$). Assume $d \in \mathbb{N}$. Then

$$\frac{a}{b} = \frac{x_0}{y_0} = \frac{d x_1}{d y_1} = \frac{x_1}{y_1}$$

for

But $x_0 > x_1$, and $x_1 \in S$.

This contradicts that x_0 is smallest element of S $\blacksquare$

So, $\frac{a}{b} = \frac{x_0}{y_0}$ and x_0, y_0 are coprime, as desired. $\blacksquare$

3 An alternative proof

__Thm:__ $\sqrt{2}$ is irrational.

__Coprime:__ $x, y \in \mathbb{Z}$ are coprime if their only common divisors are ± 1.

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Lemma: If $W = \{a + b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Z}\}$ and $x, y \in W$ then $xy \in W$ and $x + y \in W$.

Pf: Exercise!

$\mathbb{R}$ = real numbers

$= \{e, \pi, \sqrt{2}, 3, \dots\}$

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Pf of thm: Let $\alpha = \sqrt{2} - 1 \in W$.

Then, $0 < \alpha < 1$. So $\lim_{n \to \infty} \alpha^n = 0$.

But for all n $\alpha^n \in W$. So,

$\alpha^n = c + d\sqrt{2}$, $c, d \in \mathbb{Z}$. Assume $\sqrt{2} = \frac{a}{b}$

w/ $a, b \in \mathbb{Z}$. Then

$$\alpha^n = c + d\sqrt{2} = c + d\frac{a}{b} = \frac{bc + ad}{b}$$

Assume $b > 0$. So $bc + ad > 0$, So $\frac{bc + ad}{b} \geq \frac{1}{b}$.

So $\lim_{n \to \infty} \alpha^n \geq \frac{1}{b}$. Contradiction

$\{x \in S : \underline{\quad} \}$

$= \{$ All elements x of S satisfying $\underline{\quad} \}$