THE COHOMOLOGY OF SHIMURA VARIETIES AND THE LANGLANDS CORRESPONDENCE

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Disclaimer

These are greatly extended notes based on talks the author has given surrounding his recent work in [BMY] with A. Bertoloni Meli. This note is certainly very rough and that should be kept in mind while reading it. In particular, the discussion in Part II is woefully glib and is lacking in the correct attributions to the work of many brilliant mathematicians. Do not take this as a full, or even accurate, account of the history of the material discussed. If you think there are glaring errors in Part II, please feel free to contact the author so that he can fix them.

Also, Part I is a motivation for the Langlands program from the perspective of the cohomology of Shimura varieties. There is certainly no royal road to the statement of the Langlands program, and in particular Part I was written with the audience (mostly algebraic geometers) in mind. It completely ignores much of the representation theoretic and/or analytic motivation for the Langlands program. For this the author suggests that one consults the excellent article [Gel84]. Part I is also meant only to be a taster and the author has ommitted most proofs as well as certainly making intended/unintdended errors.

1. Part I: Matushima's formula and the Langlands program

1.1. **Motivation.** The goal of this note is to discuss ongoing work of the author and A. Bertoloni Meli. This work concerns the Scholze–Shin conjecture which in turn concerns the local Langlands conjecture. We would like to spend the first part of this talk trying to motivate the Langlands conjectures in the way that the author has always (personally) found most insightful.

The conjectures at large Instead of trying to motivate the local Langlands conjectures directly it is perhaps easier to try and get a grip on its global counterpart:

Conjecture 1.1 (Langlands, Buzzard-Gee, ...). *Let G be a reductive group over $\mathbb{Q}$. Fix a prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$. Then, there is commuting diagram of associations*

$$\begin{array}{ccc} \left\{\begin{matrix}\text{Algebraic automorphic}\\ \text{representations of } G\end{matrix}\right\} & \xrightarrow{\mathsf{GLC}} & \left\{\begin{matrix}\text{Algebraic admissible}\\ \text{homomorphisms } \Gamma_{\mathbb{Q}} \to {}^L G(\overline{\mathbb{Q}_\ell})\end{matrix}\right\} \\ \Big\downarrow{\scriptstyle \pi\mapsto\pi_v} & & \Big\downarrow{\scriptstyle \sigma\mapsto\sigma|_{\Gamma_{\mathbb{Q}_v}}} \\ \left\{\begin{matrix}\text{Admissible representations}\\ \text{of } G(\mathbb{Q}_v)\end{matrix}\right\} & \xrightarrow[\mathsf{LLC}]{} & \left\{\begin{matrix}\text{Admissible homomorphisms}\\ \Gamma_{\mathbb{Q}_v} \to {}^L G(\overline{\mathbb{Q}_\ell})\end{matrix}\right\} \end{array} \tag{1}$$

where v ranges over all places of $\mathbb{Q}$ except $v=\ell$.

Remark 1.2. This is, in fact, *not really* the global Langlands conjecture for G. The 'true' global Langlands conjecture requires the existence of a 'global Langlands group' $\mathscr{L}_{\mathbb{Q}}$ whose mere existence is even conjectural. The above conjecture should be a consequence of the global Langlands conjecture for G but one whose statement doesn't need to assume the existence of $\mathscr{L}_{\mathbb{Q}}$. To see why this conjecture is far from ideal (contrasting with the 'true' Langlands conjecture) we point out that the admissible homomorphism $\mathsf{GLC}(\pi)$ isn't even uniquely characterized by the property that $\mathsf{GLC}(\pi)\mid_{W_{\mathbb{Q}_p}}=\mathsf{LLC}(\pi_p)$ for almost all unramified p (even in some cases where G is a non-split torus)!

For more information on the above conjecture, including an explanation of terminology and its relation to the 'true' global Langlands conjecture, see [BG14].

Before we launch in to the 'what' and the 'how' of these conjectures, it's first useful to understand the 'why'. Namely, before we try to explain what these terms mean roughly and why such a correspondence is not unbelievable we'd like to provide the reader with three lists that will hopefully ground them when thinking about these conjectures.

We start with reasons to care (i.e. applications):

- The Langlands conjectures were used to prove Fermat's Last Theorem. (Wiles)
- The Langlands conjectures are used to study the Sato-Tate conjecture for elliptic curves. (Clozel, Harris, Shepherd-Baron,Taylor)
- The Langlands conjectures were used to prove Ramanujan's τ-conjecture. (Deligne)
- The Langlands conjectures imply Artin's L-function conjecture. (Langlands)
- The Langlands conjectures were used to show that the continuous homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\to\mathrm{Gal}(\mathbb{Q}_{\{p,\ell\}}/\mathbb{Q})$ is injective, where ℓ and p are distrinct primes and $\mathbb{Q}_{\{p,\ell\}}$ is the maximal extension of $\mathbb{Q}$ unramified outside of p and ℓ. (Chenevier–Clozel)
- The Langlands conjectures (for function fields) were used to prove finiteness of geometric representations. (Litt)

We then move on to reasons to believe:

- Many examples where the conjectures are known to be true.
- Converse theory (nice L-functions are automorphic).
- Langlands work on Eisenstein series (his letter to Weil).
- The cohomology of Shimura varieties (also a reason to care).

Finally we move on to known cases:

- The local Langlands conjecture is known for the groups $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$. (Harris–Taylor, Henniart, Scholze)
- The local Langlands conjecture is known for the symplectic groups $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{Sp}_{2n}$, the odd orthogonal $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{SO}_{2n+1}$, and the even orthogonal groups $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{SO}_{2n}$. (Arthur)

- The local Langlands conjecture is known for the quasi-split unitary groups $\mathrm{Res}_{F/\mathbb{Q}_p} U_{E/F}(N)^*$. (Mok)
- The local Langlands conjecture is known for inner forms of quasi-split unitary groups. (Kaletha–Minguez–Shin–White)
- The local Langlands conjecture is known for $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GSp}_4$. (Gan–Takeda)
- Many cases of the global Langlands conjecture are known for the groups $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$. (Harris–Lan–Taylor–Thorne, Shin, Scholze, Clozel, Harris–Taylor, Taylor–Yoshida, ...)

Moral of the above lists is that the Langlands conjecture is not some uknowable beast. There are applications of it. There are known cases of it. There are natural reasons to believe it holds true in general.

What? How? Now that we have gotten the above out of the way, let us return to the actual statement of Conjecture 1.1. It is not intended to be comprehensible except in broad strokes. Namely, what I want its statement to elicit is the feeling that there should be some association

$$\left\{\begin{matrix}\text{Automorphic representations}\\ \text{of } G\end{matrix}\right\} \to \{\text{Galois representations}\} \tag{2}$$

The following questions then present themselves to us:

Question 1.3.

(1) *What* are *automorphic representations of G?*
(2) *Why should I expect to be able to associate Galois representations to them?*

We will spend the rest of Part I trying to partially answer these questions. Our explanation is far from explaining the precise nature of the Langlands conjecture but will hopefully convince the reader that automorphic representations are not so crazy, and it's not so crazy that they should have interesting interaction with Galois representations.

Before we embark on this journey, we mention the underlying guiding principle that we will try and utilize. Suppose that we have two abstract groups H and Γ and we wish to form an association

$$\left\{\begin{matrix}\text{Representations}\\ \text{of } H\end{matrix}\right\} \to \left\{\begin{matrix}\text{Representations}\\ \text{of } \Gamma\end{matrix}\right\} \tag{3}$$

Now, if one wants two friends to dance, the first step is to get them to show up at the same party. In particular, to create such an association we would like a vector space V on which both H and Γ act. If we can find such a V satisfying two properties:

(1) The actions of H and Γ on V commute (or equivalently we have an action of $H \times \Gamma$ on V).
(2) The group H acts semisimply on V.

then we can actually create an association as in Equation (3). Namely, since H acts semisimply on V we have a decomposition

$$V = \bigoplus_{\sigma} V(\sigma) \tag{4}$$

where σ ranges over irreducible representations of H and $V(\sigma)$ is the σ-isotypic component of V. But, since Γ commutes with the action of H it's easy to see that Γ must stabilize $V(\sigma)$ or, in other words, $V(\sigma)$ is a Γ-representation. Thus, we obtain an association of a representation σ of H to a representation $V(\sigma)$ of Γ.

Now automorphic representations of G are representations of $G(\mathbb{A})$ (the adelic points of G) and Galois representations are, well, representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus to apply the above general principle we will want to find a vector space on which $G(\mathbb{A}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or something closely related) acts.

This is the perfect time to pose a question which is seemingly completely unrelated (at this point one gives a knowing wink to the audience) to the Langlands program:

Question 1.4. *How does one understand the singular cohomology of (locally) symmetric spaces?*

1.2. **Locally symmetric spaces.** In this subsection we briefly recall the definitions of locally symmetric spaces.

Begin by recalling that if H is a connected semisimple real Lie group then one has the following well-known theorem:

Theorem 1.5 (Cartain-Iwasawa-Malcev)**.** *The group H has a unique (up to conjugacy) maximal compact subgroup C and $N_H(C) = C$.*

Proof. One can see [Con14, Theorem D.2.8] and the references therein. □

> **Remark 1.6.** As loc. cit. shows, this is true even if H is just assumed to be a Lie group with finitely many connected components. In particular, if G is a reductive group over $\mathbb{R}$ then $G(\mathbb{R})$ has a unique (up to conjugacy) maximal compact subgroup (note that the fact that $G(\mathbb{R})$ has only finitely many components is [Mil17, Corollary 25.55]). This is a fact that will come up implicitly in what follows as we shall soon consider disconnected groups.

From this theorem we see that the real manifold H/C can be interpreted as the space of maximal compact subgroups of H. This space of maximal compact subgroups often times has a surprisingly interesting interpretation. We give some examples and non-examples to this point:

Example 1.7. If $H = \mathrm{SL}_2(\mathbb{R})$ and $C = \mathrm{SO}_2(\mathbb{R})$ then H/C can be identified with the upper-half plane

$$\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \tag{5}$$

by considering the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{h}$ by fractional linear transformations.

Moreover, $\mathfrak{h}$ can be thought of as a moduli space of elliptic curves with (oriented) trivialization of their homology. More precisely, let

$$\mathcal{F} : \mathsf{An} \to \mathsf{Set} \tag{6}$$

be the functor (where An denotes the category of complex analytic spaces) which associates to S the isomorphism classes of pairs $(\mathcal{E}, \psi)$ where:

- $f : \mathcal{E} \to S$ is an elliptic curve (e.g. a smooth proper map with a section whose fibers are all genus 1 curves).

- ψ is an isomorphism $\underline{\mathbb{Z}}^2 \xrightarrow{\approx} (R^1 f_* \underline{\mathbb{Z}})^\vee$ which is 'i-oriented'.

then $\mathfrak{h}$ represents $\mathcal{F}$ with representing object the elliptic curve $\mathcal{E}^{\mathrm{univ}} \to \mathfrak{h}$ such that for all $\tau \in \mathfrak{h}$ we have that $\mathcal{E}^{\mathrm{univ}}_\tau \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. For more information see [Con, §6].

Example 1.8. If $H = \mathrm{SL}_n(\mathbb{C})$ and $C = \mathrm{SU}_n(\mathbb{R})$ and we can identify H/C with the space of positive-definite Hermitian matrices:

$$H/C = \left\{ A \in \mathrm{Mat}_n(\mathbb{C}) : \begin{matrix} (1) & A = A^* \\ (2) & A > 0 \\ (3) & \det(A) = 1 \end{matrix} \right\} \tag{7}$$

Example 1.9. If H is the 1-units in $\mathbb{H}$, where $\mathbb{H}$ is the Hamiltonian quaternions, then $C = H$ and thus H/C is a point.

Remark 1.10. The truly interesting example in the above list is Example 1.7. In fact, this example indicates why such spaces H/C (or closely related ones) might be interesting to a geometer in general. Namely, recall that an elliptic curve over an analytic space S is equivalent to the structure of a polarizable variation of Hodge structure of type $\{(-1,0),(0,-1)\}$ on a rank 2 $\mathbb{Q}$-local system. One can then see Example 1.7 as part of the larger theory of *Hermitian symmetric domains* which, essentially, are period spaces for Hodge structures (with some conditions) and which are all (essentially) of the form H/C. For more information see the nice note [LZ].

Now, while these spaces H/C often times have interesting interpretations, they are topologically non-interesting:

Theorem 1.11 (Cartan)**.** *The space H/C is diffeomorphic to $\mathbb{R}^n$ where $n = \dim(H/C)$.*

Proof. Again, this follows from [Con14, Theorem D.2.8]. $\square$

So, while the spaces H/C are not topologically interesting there is a fairly natural operation that will produce interesting spaces. Namely, to get topologically interesting spaces we could quotient the spaces H/C by certain discrete Γ subgroups of their automorphism groups $\mathrm{Aut}(H/C)$ (here 'Aut' is, a priori, taken to be the group of self-diffeomorphisms of H/C).

Example 1.12. If we are in the context of Example 1.7 then a natural candidate for Γ would be the image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Aut}(\mathfrak{h})$. More generally, we could think of (the image in $\mathrm{Aut}(\mathfrak{h})$ of) certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$ like

$$\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})) \tag{8}$$

for $N \geqslant 1$ an integer.

Example 1.13. If we are in the context of Example 1.8 we might take the image in $\mathrm{SL}_2(\mathbb{C})$ of subgroups of the form $\mathrm{SL}_2(\mathbb{Z}[i])$ or $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ (or subgroups of these constructed as in Example 1.12).

Example 1.14. If we are in the context of Example 1.9 then there is nothing to do.

In all examples we see that we constructed our subgroups of $\mathrm{Aut}(H/C)$ as discrete subgroups Γ of H constructed by considering $H = G(\mathbb{R})^+$ for $G/\mathbb{Q}$ an algebraic group and taking Γ to be certain discrete subgroups of $G(\mathbb{Q})$.

This suggests to us the following deifnition:

Definition 1.15. *Let G be a semisimple group over $\mathbb{Q}$. A subgroup $\Gamma \subseteq G(\mathbb{Q})$ is called* congruence *if there exists an embedding $G \hookrightarrow \mathrm{GL}_n$ such that Γ contains*

$$\ker(\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})) \cap G(\mathbb{Q}) \tag{9}$$

for some $N \geqslant 1$. Set $X_G^+ = X^+ := G(\mathbb{R})^+/C$. Then, a space of the form $\Gamma^+ \backslash X^+$ is called a symmetric space *for G (here $\Gamma^+ = \Gamma \cap G(\mathbb{R})^+$).*

Remark 1.16. What does 'space' mean in the above definition of symmetric space? In general $\Gamma^+ \backslash X^+$ is only an orbifold as Γ may have torsion, and so may not act properly discontinuously on X. But, for all Γ that are sufficiently small (e.g. those contained in a group of the form in Equation (9) for $N \geqslant 3$) so as to be 'neat' (which one should think of as the minimal modification to the definition of 'torsion free' that behaves well functorially) Γ^+ acts properly discontinuously on X^+ and thus $\Gamma^+ \backslash X^+$ is a smooth manifold. Working with neat subgroups suffices in all cases of interest to us, so we shall always assume (unless stated othewise) that our Γ^+ are neat but we don't explicitly state this.

Remark 1.17. The above definition is, in some sense, non-standard and should be taken only as motivational tool. It is really the objects of Definition 1.24 that are the 'correct' objects to consider.

Let us give some examples of symmetric spaces:

Example 1.18. If $G = \mathrm{SL}_2$ and $\Gamma = \Gamma(N)$ as in Equation (8) (for $N \geqslant 3$) then $\Gamma(N)^+ \backslash X^+_{\mathrm{SL}_2} =: Y^+(N)$ has particularly interesting meaning. Namely, let us fix an isomorphism $\iota : \mu_N(\mathbb{C}) \cong \mathbb{Z}/N\mathbb{Z}$. Then, $Y^+(N)$ represents the functor

$$\mathcal{F}_N^\iota : \mathsf{An} \to \mathsf{Set} \tag{10}$$

which associates to S the isomorphism classes of pairs $(\mathcal{E}, \psi_N^\iota)$ where

- $f : \mathcal{E} \to S$ is an elliptic curve.
- $\psi_N^\iota : \mathcal{E}[N] \xrightarrow{\approx} \underline{(\mathbb{Z}/N\mathbb{Z})^2}$ is an isomorphism of sheaves such that the induced isomorphism $\det(\psi_N^\iota) : \mu_{N,S} \to \underline{\mathbb{Z}/N\mathbb{Z}}$ is identified with ι (note that we have used the fact that $\wedge^2 \mathcal{E}[N] \cong \mu_{N,S}$ by the Weil pairing). This isomorphism ψ_N^ι is called a $\Gamma(N)$-*level structure*.

This complex analytic space $Y^+(N)$ (inheriting its complex structure from $\mathfrak{h}$) is actually a smooth connected affine curve. We denote its smooth algebraic compactification by $X^+(N)$ (don't confuse this X with the X for symmetric spaces—just an unfortunate notational clash). For more information see [Con, §8].

To see that symmetric spaces really do heavily depend on the choice of a model of $G_\mathbb{R}$ over $\mathbb{Q}$ (or equivalently what subgroups of $\mathrm{Aut}(X^+)$ we are allowed to quotient by) we note the following:

Example 1.19. Let D be a division algebra over $\mathbb{Q}$ of dimension 4 which is split over $\mathbb{R}$ (e.g. the quaternion algebra $Q(2,3)$). Take G to be the group over $\mathbb{Q}$ with $\mathbb{Q}$-points given by the elements of $D^\times$ of reduced norm 1. Then, $G_\mathbb{R} \cong \mathrm{SL}_{2,\mathbb{R}}$ so $X_G^+ \cong X_{\mathrm{SL}_2}^+$. But, the symmetric spaces for G are very different those of SL_2. In fact, for every congruence subgroup Γ of $G(\mathbb{Q})$ the symmetric space $\Gamma^+\backslash X_G^+$ is a compact connected Riemann surface (thus a proper complex curve). This is in stark contrast to the symmetric spaces of SL_2 which have affine symmetric spaces. Such spaces are called *Shimura curves* and they don't have explicit moduli interpretations.

In fact, one has the following result of Borel:

Theorem 1.20 (Borel)**.** *Let G be a semisimple group over $\mathbb{Q}$. Then, every (equivalently one) symmetric space for G is compact if and only if there exists* <u>no</u> *embedding of algebraic groups $\mathbb{G}_{m,\mathbb{Q}} \hookrightarrow G$.*

Remark 1.21. See Remark 1.33 to get a handle on why this is true.

Since it plays such a pivotal role in this note let us formalize this non-embedding property:

Definition 1.22. *An algebraic group G over a field F is called F-*anisotropic *if there exists no embedding $\mathbb{G}_{m,F} \hookrightarrow G$.*

Now, in the above examples there were many annoyances related to having to worry about certain connected components. While a lot of these are just that, annoyances, some pose serious obstacles to studying the hidden arithmetic associated to symmetric spaces.

Consider the following:

Example 1.23. Consider the spaces $Y^+(N)$ from Example 1.12. Note that the moduli problem for $Y^+(N)$ is tantalizing close to being a moduli problem definable over $\mathbb{Q}$ (or even $\mathbb{Z}[\frac{1}{N}]$) save for this issue about having to choose an identification $\iota : \mu_N(\mathbb{C}) \cong \mathbb{Z}/N\mathbb{Z}$. In fact, as it stands, the spaces $Y^+(N)$ are only defined over $\mathrm{Spec}(\mathbb{Q}(\mu_N(\mathbb{C}))$ (since we need at least one identification $\mu_n \cong \mathbb{Z}/N\mathbb{Z}$ over our ground field). Now for a fixed N this is not a huge issue. That said, we see that there are natural projection maps $Y^+(NM) \to Y^+(N)$ (essentially because a $\Gamma(MN)$ level structure begets a $\Gamma(N)$ level structure) for any $M \geqslant 1$ and these maps will only be definable over $\mathrm{Spec}(\mathbb{Q}(\mu_{\mathrm{lcm}(M,N)}(\mathbb{C})))$. So while each $Y^+(N)$ is defined over a number field, the entire system $\{Y^+(N)\}$ is only defined over $\overline{\mathbb{Q}}$ which becomes more problematic if our goal is to be studying arithmetic.

To fix this we note that for each N if instead of $Y^+(N)$ we consider

$$Y(N) := \coprod_{\iota:\mu_N(\mathbb{C})\cong\mathbb{Z}/N\mathbb{Z}} Y^+(N) \tag{11}$$

(where we are identifying the ι^{th}-copy of $Y^+(N)$ as having moduli problem $\mathcal{F}_N^\iota$) this problem disappears. Namely, $Y(N)$ represents the moduli problem

$$\mathcal{F}_N : \mathsf{An} \to \mathsf{Set} \tag{12}$$

associating to S isomorphism classes of pairs $(\mathcal{E}, \psi)$ where

- $f : \mathcal{E} \to S$ is an elliptic curve.
- $\psi : \mathcal{E}[N] \xrightarrow{\approx} \underline{(\mathbb{Z}/N\mathbb{Z})^2}$ is an isomorphism.

In particular, we see that this moduli problem makes sense even over $\mathbb{Z}[\frac{1}{N}]$ (or $\mathbb{Z}$ but obviously it will have empty fibers over $\mathbb{F}_p$ for any $p \mid N$ and thus will not be flat) and thus $Y(N)$ actually has a model over $\mathbb{Q}$ (e.g. by flat descent for affine morphisms—more rigorously one can give a way overkill proof of this by combining [Sta18, Tag04SK] and [Sta18, Tag03WG]). In fact, we see that the system $\{Y(N)\}$ as a whole has a model over $\mathbb{Q}$. We denote by $X(N)$ the smooth proper algebraic model of $Y(N)$.

As the above example indicates restricting ourselves to connected objects highly obfuscates the arithmetic properties of symmetric spaces. Thus, we would like a way to consider not only symmetric spaces $\Gamma^+ \backslash X^+$ but also disjoint unions $\bigsqcup_i \Gamma_i^+ \backslash X^+$. Of course, the connected components of these disjoint unions should not be random—they should have something to do with one another (as in the case of Example 1.23).

It turns out that, with a fair bit of insight, the correct way to frame this discussion is *adelically*. Namely, we have the following definition:

Definition 1.24. *Let G be a reductive group over $\mathbb{Q}$. Let us fix a maximal compact subgroup $K_\infty^{\max}$ of $G(\mathbb{R})$ and let $C := (K_\infty^{\max})^\circ$. Let us then fix/denote:*

- *K_∞ to be a compact subgroup of $G(\mathbb{R})$ such that $C \subseteq K_\infty \subseteq K_\infty^{\max}$*
- *A_G is the maximal $\mathbb{Q}$-split subtorus of $Z(G)$.*

Then, for any compact open subgroup $K_f \subseteq G(\mathbb{A}_f)$ we set

$$S_G(K_f) := G(\mathbb{Q}) \backslash G(\mathbb{A}) / (K_\infty K_f A_G(\mathbb{R})^+) \tag{13}$$

A space of the form $S_G(K_f)$ is called a locally symmetric space *for G.*

Remark 1.25. Here we are denoting by $\mathbb{A} = (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \times \mathbb{R}$ the adele ring for $\mathbb{Q}$ and by $\mathbb{A}_f$ its subring of finite adeles $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. We topologize this by the restricted direct topology and topologize $G(\mathbb{A})$ and $G(\mathbb{A}_f)$ in the usual way (e.g. see [Con12]).

Remark 1.26. It would perhaps be more accurate to denote these symmetric spaces by $S_G(K_\infty, K_f)$ or something to denote the dependence on K_∞, but we will not do so. This extra choice of having this K_∞ being an arbitrary compact subgroup of $G(\mathbb{R})$ containing the group C of $G(\mathbb{R})^+$ will turn out to be very convenient. But, more often than not, K_∞ can be taken to be C.

While this definition, with the addition of the adele ring of $\mathbb{Q}$, is quite jarring at first it really is no more than the correct formalism for talking about "'disjoint unions of symmetric spaces for G with respect to 'similar groups"'. To convince the reader of this, we state the following fact (whose proof we leave to the reader—see [PS92, Chapter 9]):

Fact 1.27. *The set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$ is finite. Let $g_1, \ldots, g_k$ be double coset representatives. Then,*

$$S_G(K_f) = \bigsqcup_{j=1}^{k} \Gamma_j \backslash X \tag{14}$$

where $\Gamma_j := G(\mathbb{Q}) \cap g_j K_f g_j^{-1}$ and $X := A_G(\mathbb{R})^+ \backslash G(\mathbb{R})/K_\infty$. Moreover, $\Gamma_j \backslash X$ is a disjoint union of spaces isomorphic to Γ_j^+/X^+ where $\Gamma_j^+ \backslash X^+$ is a symmetric space for $G^{\mathrm{der}}(\mathbb{R})^+$.

Remark 1.28. To further reinforce that the groups $G(\mathbb{Q}) \cap g_j K_f g_j^{-1}$ are not that crazy one can try and solve the following exercise: a subgroup $\Gamma \subseteq G(\mathbb{Q})$ is congruence if and only if $\Gamma = K_f \cap G(\mathbb{Q})$ where K_f is a compact open subgroup of $G(\mathbb{A}_f)$. This gives a definition of congruence that is inherently independent of the choice of a faithful representation of G.

Remark 1.29. The reader might be curious about the inclusion of this term $A_G(\mathbb{R})^+$ in the above quotient. The inclusion of this to the quotient is largely a matter of convenience (as it simplifies the representation theory) and is topologically irrelevant as one can show that the removal of $A_G(\mathbb{R})^+$ in the quotient just adds some junk components of the form $\mathbb{R}^d$.

Let us give an example:

Example 1.30. Let us take $G = \mathrm{GL}_2$. Note then that A_G is just the diagonal matrices in GL_2 and for K_∞ we take $\mathrm{SO}_2(\mathbb{R})$. Note then that

$$X = \mathbb{R}^{>0} \backslash \mathrm{GL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \tag{15}$$

can be identified with the double half-plane

$$\mathfrak{h}^{\pm} = \mathbb{C} - \mathbb{R} \tag{16}$$

where $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathfrak{h}^{\pm}$ by fractional linear transformations. If we then set

$$K(N) := \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})) \tag{17}$$

then we leave the reader to show that $S_{\mathrm{GL}_2}(K(N)) = Y(N)$.

Remark 1.31. Again if we don't assume that K_f is a 'neat' compact open subgroup of $G(\mathbb{A}_f)$ then the space $S_G(K_f)$ is really better thought of an orbifold (instead of a real manifold) much as in Remark 1.16 and so, unless stated otherwise, we will always implicitly assume that K_f is neat. In practice this is a non-issue since we often care only about $S_G(K_f)$ for a cofinal system of compact open subgroups (e.g. see Equation 23) of $G(\mathbb{A}_f)$ and neat subgroups form such a cofinal system.

The analogue of Borel's theorem (Theorem 1.20) in this context is:

Theorem 1.32 (Borel)**.** *Let G be a reductive group over $\mathbb{Q}$. Then, the space $S_{K_f}(G)$ is compact for some K_f (equivalently for all K_f) if and only G^{der} is $\mathbb{Q}$-anisotropic.*

Remark 1.33. For those interested, the reason for this is actually quite nice. Namely, there is a natural compactification of symmetric spaces for G called the *Baily-Borel compactification* (e.g. see Goresky's chapter in [AEK05]). The boundary components of this compactification are indexed by the proper parabolic subgroups P of G (i.e. the proper closed subgroups P of G such that the fppf quotient G/P is proper). So, we see that the symmetric spaces are compact if and only if G has no proper parabolic subgroups. This turns out to be equivalent to the claim that G^{der} is $\mathbb{Q}$-anisotropic (although this equivalence requires a non-trivial amount of work).

Let us note that if $K_f^1 \supseteq g^{-1}K_f^2 g$ is a containment of compact open subgroups of $G(\mathbb{A}_f)$, where $g \in G(\mathbb{A}_f)$, then there is a natural covering map of smooth manifolds

$$[g] : S_{K_f^1}(G) \to S_{K_f^2} \tag{18}$$

(thought of as 'projection to a double coset by a larger group then right action by g') this assembles to an action of $G(\mathbb{A}_f)$ on

$$S_G := \varprojlim S_G(K_f) \approx G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)) \tag{19}$$

where $G(\mathbb{Q})$ is acting diagonally on this last term and this isomorphism is as $G(\mathbb{A}_f)$-spaces (e.g. see [Roh96, Proposition 1.9]). We call this action of $G(\mathbb{A}_f)$ on S_G the *Hecke action* (see Remark 1.43 for the relation to a perhaps more familiar relationship notion of Hecke action).

Remark 1.34. Unless the author is mistaken (which is wholly possible) there is some disagreement between [Roh96, Proposition 1.9] and [Mil04, Theorem 5.28]. In all instances of interest to us we will assume that $(A_G)_\mathbb{R}$ is the largest $\mathbb{R}$-split subtorus of $G_\mathbb{R}$ (i.e. that $(A_G)_\mathbb{R} = A_{G_\mathbb{R}}$) in which case there is no dissonance. If the reader would like to assume this equality $(A_G)_\mathbb{R} = A_{G_\mathbb{R}}$ they should feel free to do so or, perhaps even better, just take S_G to be the limit $\varprojlim S_G(K_f)$ and understand that it's 'approximately' $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$.

1.3. **The cohomology of locally symmetric spaces.**

We are now interested in understanding the singular cohomology $H^i_{\mathrm{sing}}(S_G(K_f), \mathbb{Q})$ or, more generally, the cohomology of certain local systems on $S_G(K_f)$. Given the complicated nature of Example 1.23 and Example 1.19 this is likely to be a non-trivial task.

Let us begin by defining the local systems $\mathscr{F}_\xi$ that will be of interest to us. Namely, we have the following:

Definition 1.35. *Let $\xi : G_L \to \mathrm{GL}(V)$ be an algebraic representation where $L/\mathbb{Q}$ is finite (we call such an object an* algebraic $\overline{\mathbb{Q}}$-representation *of G). Let us define an L-local system $\mathscr{F}_{\xi,K_f}$ (or just $\mathscr{F}_\xi$ if K_f is clear from context) on $S_G(K_f)$ as the sheaf of sections associated to the map*

$$G(\mathbb{Q})\backslash V \times X \times G(\mathbb{A}_f)/K_f \to S_G(K_f) \tag{20}$$

where $G(\mathbb{Q})$ acts diagonally on $V \times X \times G(\mathbb{A}_f)$ (where note that $G(\mathbb{Q}) \subseteq G(L)$ acts on V) and K_f only acts on the rightmost factor.

Remark 1.36. To have that $\mathscr{F}_{\xi,K_f}$ is well-behaved (e.g. that $\mathscr{F}_{\xi,K_f,x} \cong V$ for $x \in S_{K_f}(G)$) we need to assume that $Z(\mathbb{Q}) \cap K_f \subseteq \ker \xi(\mathbb{Q})$. If $(A_G)_\mathbb{R} = A_{G_\mathbb{R}}$ (as in Remark 1.34) this is automatic for K_f sufficiently small (which is all that is of interest to us as well) and so this condition is ignorable in practice, thus we will not belabor this condition—but it should be pointed out.

Example 1.37. If ξ is taken to be the trivial representation of G then it's easy to see that $\mathscr{F}_{\xi,K_f} = \underline{L}$ for all compact open $K_f \subseteq G(\mathbb{A}_f)$.

We can then rephrase Question 1.4 in the more precise way:

Question 1.38. *Let G be a reductive group over $\mathbb{Q}$ and ξ an algebraic $\overline{\mathbb{Q}}$-representation of G. Then how do we describe $H^i(S_G(K_f), \mathscr{F}_\xi)$?*

Remark 1.39. Let us note that if we give a decomposition

$$S_G(K_f) = \bigsqcup_t \Gamma_t^+ \backslash X^+ \tag{21}$$

as in Fact 1.27 one has (at least if K_f is neat) that X^+ is the universal cover of $\Gamma_t^+\backslash X^+$ and (by Theorem 1.11) and that it's contractible. In other words, we see that $\Gamma_t^+\backslash X^+$ is a topological $K(\Gamma_t^+, 1)$ and so if $\xi : G_L \to \mathrm{GL}(V)$ then

$$H^i(\Gamma_t^+\backslash X^+, \mathscr{F}_\xi) = H^i(\Gamma_t^+, V) \tag{22}$$

so, in essence, computing the cohomology of $S_G(K_f)$ comes down to computing group cohomology (this doesn't really make the question any easier!).

Let us be more precise about in what way we want to describe the cohomology groups $H^i(S_G(K_f), \mathscr{F}_\xi)$. Indeed, let us begin by defining the group

$$H^i(S_G, \mathscr{F}_\xi) := \varinjlim_{K_f} H^i(S_G(K_f), \mathscr{F}_\xi) \tag{23}$$

(where this limit makes sense since the pullback of $\mathscr{F}_{\xi,K_f^2}$ along the map in Equation (18) is canonically $\mathscr{F}_{\xi,K_f^1}$) has a natural action of $G(\mathbb{A}_f)$ (see [Roh96] for alternative ways to describe this colimit terms of honest group cohomology/sheaf cohomology).

In fact, note that for any fixed $K_f \subseteq G(\mathbb{A}_f)$ compact open we actually have, by the Hocschild-Serre spectral sequence, a natural identification

$$H^i(S_G, \mathscr{F}_\xi)^{K_f} \cong H^i(S_G(K_f), \mathscr{F}_\xi) \tag{24}$$

This indicates that the there is no direct action of $G(\mathbb{A}_f)$ on $H^i(S_G(K_f), \mathscr{F}_\xi)$ for a fixed K_f.

But, this situation is a familiar (depending on your mathematical bend) and fixable one. Namely, we have the following well-known definition:

Definition 1.40. *The* Hecke algebra *of $G(\mathbb{A}_f)$ (with coefficients in L), denoted $\mathscr{H}_L(G(\mathbb{A}_f))$ is the set of locally constant compactly supported functions $f : G(\mathbb{A}_f) \to L$. It's a (usually* *non-unital) L-algebra under the convolution operator*

$$(f_1 * f_2)(g) := \int_{G(\mathbb{A}_f)} f_1(h^{-1}g)f_2(g)\,dh \tag{25}$$

For any compact open subgroup $K_f \subseteq G(\mathbb{A}_f)$ we denote by $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$ the subalgebra of $\mathscr{H}_L(G(\mathbb{A}_f))$ consisting of K_f-biinvariant functions. This is a unital algebra with unit $e_{K_f} := \mathrm{vol}(K_f)^{-1}\mathbb{1}_{K_f}$. It is spanned (as a vector L-space) by the functions $\mathbb{1}_{K_f g K_f}$ for $g \in G(\mathbb{A}_f)$.

Remark 1.41. In the above, and throughout this note, we are supressing the surprisingly subtle notion of choosing a 'good' Haar measure on the locally compact Hausdorff $G(\mathbb{A}_f)$. See [HG, §3.5] for details on this topic.

The reason that this is a useful concept is the following:

Fact 1.42. *Let M be a smooth L-representation (where L can be any characteristic 0 field) of $G(\mathbb{A}_f)$ (smoothness means that we have an equality of sets $M = \bigcup_{\substack{K_f \subseteq G(\mathbb{A}_f) \\ \text{compact open}}} M^{K_f}$) then M is a $\mathscr{H}_L(G(\mathbb{A}_f))$-module with operation*

$$f \cdot m := \int_{G(\mathbb{A}_f)} f(h)(h \cdot m)\,dh \tag{26}$$

which is non-degenerate (i.e. for every element $m \in M$ there exists $f \in \mathscr{H}_L(G(\mathbb{A}_f))$ such that $f \cdot m = m$). Moreover, this association is an equivalence of categories

$$\left\{\begin{matrix}\text{Smooth } L\text{-representations} \\ \text{of } G(\mathbb{A}_f)\end{matrix}\right\} \to \left\{\begin{matrix}\text{Non-degenerate} \\ \mathscr{H}_L(G(\mathbb{A}_f))\text{-representations}\end{matrix}\right\} \tag{27}$$

Thus, in particular, the Hecke action of $G(\mathbb{A}_f)$ on $H^i(S_G, \mathscr{F}_\xi)$ is 'equivalent' to the action of the Hecke algebra $\mathscr{H}_L(G(\mathbb{A}_f))$ on $H^i(S_G, \mathscr{F}_\xi)$. How does this observation help us fix the fact that $G(\mathbb{A}_f)$ doesn't directly act on $H^i(S_G(K_f), \mathscr{F}_\xi)$?

Well, note that for any smooth L-representation M of $G(\mathbb{A}_f)$ that:

(1) The action of e_{K_f} on M is the projection operator $M \to M^{K_f}$.
(2) The subalgebra $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$ is precisely $e_{K_f} * \mathscr{H}_L(G(\mathbb{A}_f)) * e_{K_f}$.

In particular, we see that while M^{K_f} is not stabilized by $G(\mathbb{A}_f)$ it is a $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$-module. In particular, $H^i(S_G(K_f), \mathscr{F}_\xi)$ is a $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$-module.

Remark 1.43. The action of $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$ on $H^i(S_G(K_f), \mathscr{F}_\xi)$ might have a more familiar form to the reader who has had some exposure to modular curves (or more generally Shimura varieties). Namely, for every double coset $K_f g K_f$ (whose indicator functions, as you'll recall, generate $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$) one can get a so-called *Hecke* *correspondence*

$$\begin{tikzcd} & S_G(K_f \cap gK_fg^{-1}) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ S_G(K_f) & & S_G(K_f) \end{tikzcd} \tag{28}$$

where p_1 is the projection operator and p_2 is the projection to $S_G(gK_fg^{-1})$ and then action by g (which does have image in $S_G(K_f)$). One can show that this can be upgraded to a cohomological correspondence on $\mathscr{F}_{\xi,K_f}$ and thus one gets an endomorphism of $H^i(S_G(K_f), \mathscr{F}_\xi)$ in the usual way. This endomorphism, in fact, agrees with the action of $\mathbb{1}_{K_fgK_f}$. Since, again, the elements $\mathbb{1}_{K_fgK_f}$ generate $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$ this observation allows one to interpret

$$\mathrm{im}(\mathscr{H}_L(G(\mathbb{A}_f), K_f) \to \mathrm{End}_L(H^i(S_G(K_f), \mathscr{F}_\xi)) \tag{29}$$

as the 'subalgebra of cohomology operations coming from Hecke correspondences'.

So, with the above set-up, the next refinement of Question 1.38 is then the following:

Question 1.44. *Let G be a reductive group over $\mathbb{Q}$ and ξ an algebraic $\overline{\mathbb{Q}}$-representation of G. Then, how do we describe $H^i(S_G(K_f), \mathscr{F}_\xi)$ as a $\mathscr{H}_L(G(\mathbb{A}_f))$-module?*

To give a simple yet satisfcatory answer to this question we will almost have to assume that G^{der} is $\mathbb{Q}$-anisotropic. In other words, in many cases below we are going to restrict ourselves to the case when our locally symmetric spaces $S_G(K_f)$ are compact. This assumptions is useful in many ways:

(1) It allows for one to not have to worry about distinctions between cuspidal objects (e.g. cohomology or automorphic representations) and non-cuspidal objects.
(2) Due to the mentioned simplifications in (1) we don't need to worry very much about the distiction between 'smooth' and 'L^2' automorphic representations.
(3) It allows us to not have to worry about the Baily-Borel compactification of our locally symmetric spaces.
(4) In a same vein to (3), it allows us to work with actual singular cohomology (resp. étale cohomology) as opposed to L^2-cohomology (resp. intersection cohomology).

The astute reader will notice that this anistropocity condition is quite strong and precludes, for example, the consideration of some of the most obvious groups (e.g. $G = \mathrm{GL}_2$). This is a huge restriction when one is interested in the global Langlands conjecture but, as it turns out, is usually a non-issue if one is interested in local Langlands.

The reason for this, roughly, is that while GL_n does not satisfy this anisotropicity assumption, it is the local component of a group that does. For example:

Example 1.45. Let D be a division algebra over $\mathbb{Q}$ of dimension n^2 such that D is split at p —such objects exist by the fundamental exact sequence of class field theory (see [Mil, Theorem VIII.4.2]). Then, $G := D^\times$ is a reductive group such that G^{der} is $\mathbb{Q}$-anisotropic but $G_{\mathbb{Q}_p} = \mathrm{GL}_{n,\mathbb{Q}_p}$.

Remark 1.46. This example typifies a common theme in the study of local Langlands by global methods. Namely, relating local objects for a group G_p over $\mathbb{Q}_p$ to global objects requires choosing a group G over $\mathbb{Q}$ such that $G_{\mathbb{Q}_p} = G_p$. One can often times choose G to satisfy very powerful restrictions (e.g. some sort of anisotropicity condition) that G_p does not itself satisfy.

If we are in the situation where G^{der} is $\mathbb{Q}$-anisotropic then we have the following result without any serious caveats:

"Theorem" 1.47 (Informal Matsushima's formula)**.** *Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is $\mathbb{Q}$-anisotropic. Then, the $\mathscr{H}_L(G(\mathbb{A}_f), K_f)$-module $H^i(S_G(K_f), \mathscr{F}_\xi)$ is semisimple and its simple constitutents are (the K_f-invariants) of (the finite part of) automorphic representations of G.*

This theorem is in scare quotes to scare away anyone willing to accept this informal statement without asking what all of these terms mean! Let us now be more precise.

The first step is to define automorphic representations for G in this situation. Because we are assuming that G^{der} is $\mathbb{Q}$-anisotropic we are able to avoid many of the hairy details that usually surround such definitions:

Theorem 1.48. *Let $\chi : A_G(\mathbb{R})^+ \to \mathbb{C}^\times$ be a quasi-character. Then, the $G(\mathbb{A})$ representation*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi) := \left\{ f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C} : \begin{array}{ll} (1) & f(zg) = \chi(z) f(g) \text{for all } z \in A_G(\mathbb{R})^+ \\ (2) & f\chi^{-1} : [G] \to \mathbb{C} \text{ is measurable} \\ (3) & \displaystyle\int_{[G]} |(f\chi^{-1})(g)|^2 \, dg < \infty \end{array} \right\} \tag{30}$$

(where $(g \cdot f)(x) := f(xg)$) decomposes discretely. In other words, there exists a Hilbert space decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi) = \widehat{\bigoplus_{\pi}} m(\pi)\pi \tag{31}$$

where π runs over the irreducible unitary subrepresentations of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$ and $m(\pi)$ is an integer.

Proof. For a proof of this result see [HG, Chapter 9]. □

Here we have abbreviated $G(\mathbb{Q})\backslash G(\mathbb{A})/A_G(\mathbb{R})^+$ by $[G]$ (the so-called *adelic quotient* of G). One of the main reasons it's useful to work with $[G]$ opposed to $G(\mathbb{Q})\backslash G(\mathbb{A})$ directly is that the former has finite volume and the latter needn't (e.g. see [HG, §2.6]).

We then have the following simple definition:

Definition 1.49. *Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is $\mathbb{Q}$-anisotropic. Then, an* automorphic representation *of G (with central character χ) is an irreducible unitary representation of $G(\mathbb{A})$ occuring in the decomposition in Equation (31).*

Remark 1.50. What made this definition of automorphic representation so painless and, in particular, why it's consonant with the 'smooth' notion of automorphic representations is precisely because G^{der} is $\mathbb{Q}$-anisotropic. Namely, the operative thing is that there is an analogue of Theorem 1.32. Namely, G^{der} is $\mathbb{Q}$-anisotropic if and only if $[G]$ is compact—this makes the decomposition in Equation 31 possible. For those more familiar with automorphic forms, another way to think about this is G^{der} is $\mathbb{Q}$-anisotropic implies that every element of $L^2([G])$ is cuspidal (since there are no proper rational parabolics). For more information see [HG, Chapter 9].

Let us internalize what we have just defined. For G a reductive group over $\mathbb{Q}$ the group $G(\mathbb{A})$ is a locally compact Hausdorff group containing $G(\mathbb{Q})$ as a discrete subgroup. This setting is precisely the setting in which abstract harmonic analysis takes place. In essence, automorphic representations of G are the harmonic analytic representations of $G(\mathbb{A})$ (relative to the discrete subgroup $G(\mathbb{Q})$.

Before we state the precise version of Matsushima's formula we note the following well-known theorem of Flath:

Theorem 1.51 (Flath [Fla79])**.** *Let π be an automorphic representation of G. Then, there is associated to π a tensor product*

$$\pi_\infty \otimes \pi^\infty \tag{32}$$

where π_∞ is an irreducible $(\mathfrak{g}, K_\infty^{\max})$-module (where $\mathfrak{g} := \mathrm{Lie}(G(\mathbb{R}))$) for $G(\mathbb{R})$ and π^∞ is an irreducible smooth representation of $G(\mathbb{A}_f)$. Moreover, the choice of a smooth reductive model $\mathcal{G}$ of G over $\mathbb{Z}[\frac{1}{n}]$ induces a restricted tensor product decomposition

$$\pi^\infty = {\bigotimes_p}' \pi_p \tag{33}$$

where π_p is an irreducible smooth representation of $G(\mathbb{Q}_p)$ according to the decomposition of topological groups

$$G(\mathbb{A}_f) = {\prod_p}' G(\mathbb{Q}_p) \tag{34}$$

(as in [Con12, §3]). This decomposition is canoncially independent of the choice of $\mathcal{G}$.

Proof. See [HG, §5.7] for a discussion of this theorem and, in particular, an explication of the notation. Note that we have been careful not to write that the object in Equation (32) is equal/isomorphic to π. What is actually true is that this object is isomorphic to $\pi^{\mathrm{sm},K_{\max}}$ where $K_{\max}$ is a (non-unique, even up to conjugacy) maximal compact subgroup of $G(\mathbb{A})$ and $\pi^{\mathrm{sm},K_{\max}}$ denote the set of smooth and $K_{\max}$-finite vectors in π. □

Remark 1.52. We don't seek here to define what a $(\mathfrak{g}, K_\infty^{\max})$-module is, noting only that in rough terms it's a representation of $\mathfrak{g} \times K_\infty^{\max}$ for which the $\mathfrak{g}$-action and the $K_\infty^{\max}$-action agree on $\mathrm{Lie}(K_\infty^{\max})$. For more details see [HG, §4.4].

That said, let us make a remark about the necessity of the $(\mathfrak{g}, K_\infty^{\max})$-module in the above theorem. It would be nice if one had a decomposition as in Equation (32) where π_∞ is actually a unitary representation of $G(\mathbb{R})$. But, since the decomposition in Equation (32) is of an algebraic nature, one needs to replace such analytically minded representations (i.e. representations on Hilbert spaces) with an algebraic substitute. One should think of a $(\mathfrak{g}, K_\infty^{\max})$-module as a sort of 'algebraic model' of an analytic (i.e. unitary representation) of $G(\mathbb{R})$. To make precise sense of this see [HG, Proposition 4.4.2] and see [HG, Theorem 4.4.4] to see why this operation is not losing much information.

We now in the position to state Matsushima's formula:

Theorem 1.53 (Matsushima's formula)**.** *Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is $\mathbb{Q}$-anisotropic and let $\xi : G_L \to \mathrm{GL}(V)$ be a geometrically irreducible algebraic $\overline{\mathbb{Q}}$-representation. Then, for ι_L an embedding $L \hookrightarrow \mathbb{C}$ and all $i \geqslant 0$ there is a decomposition of $\mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f), K_f)$-modules*

$$H^i(S_G(K_f), \mathscr{F}_\xi \otimes_{\iota_L} \mathbb{C}) = \bigoplus_\pi (\pi^\infty)^{K_f} \otimes H^i(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty \otimes V_{\mathbb{C}}) \tag{35}$$

where π ranges over the automorphic representations of G which have central character $\chi := (\xi\,|_{A_G(\mathbb{R})^+})^{-1}$.

Let us note several things about this theorem:

- The $\mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f), K_f)$-action on the right-hand side of Equation (35) is only on the first factor of each summand. The second tensor factor only acts as a multiplicity term.
- The $\mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f), K_f)$-modules $(\pi^\infty)^{K_f}$ are simple (as one can easily deduce from the irreducibility of the $G(\mathbb{A}_f)$-representation π^∞). Thus, we see that Matushima's formula is telling us that $H^i(S_G(K_f), \mathscr{F}_\xi \otimes_{\iota_L} \mathbb{C})$ is a semisimple module for $\mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f), K_f)$ and Equation (35) indicates its decomposition into simple modules.
- Note that our embedding ι_L allows us to view $V \otimes_{\iota_L} \mathbb{C}$ (which we have abbreviated $V_{\mathbb{C}}$ in the above) as a $G(\mathbb{C})$-representation and thus a $G(\mathbb{R})$-representation. This then produces in the usual way (e.g. see [HG, Proposition 4.4.2]) a $(\mathfrak{g}, K_\infty^{\max})$-module structure on $V_{\mathbb{C}}$ (and thus a $(\mathfrak{g}, K_\infty)$-module structure). We then have the $(\mathfrak{g}, K_\infty)$-module $\pi_\infty \otimes V_{\mathbb{C}}$ (with the diagonal action). Since the $A_G(\mathbb{R})^+$-characters of V and π_∞ are inverses (by construction) we have that $\mathfrak{a}_G := \mathrm{Lie}(A_G(\mathbb{R})^+)$ acts trivially on $\pi_\infty \otimes V_{\mathbb{C}}$ which is why we can consider it as a $(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty)$-module (note that $K_\infty \cap A_G(\mathbb{R})^+$ is trivial and so K_∞ is still a compact subgroup of the connected component of $G(\mathbb{R})/A_G(\mathbb{R})^+$).
- This cohomology group $H^i(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty \otimes V_{\mathbb{C}})$ should be read as "the $(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty)$-cohomology of $\pi_\infty \otimes V_{\mathbb{C}}$". It is a cohomology theory with a definition much as the reader suspects (e.g. see [BW13, Chapter 1] for definitions). It is a finite dimensional $\mathbb{C}$-space.

We will not prove Theorem 1.53 citing [HG, §15.5] and the references therein. That said, the idea is remarkably simple. Namely, to compute the cohomology of the local system $\mathscr{F}_\xi \otimes_{\iota_L} \mathbb{C}$ one should, following de Rham's theorem, tensor this local system with the de Rham complex and take cohomology of the resulting de Rham-like-complex.

Writing down these spaces of differentials one quickly relates it to certain smooth functions

$$f : G(\mathbb{Q})\backslash G(\mathbb{A})/A_G(\mathbb{R})^+K_f \to \mathbb{C} \tag{36}$$

which already starts to look a lot like (the smooth and $K_{\max}$-finite vectors in) the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$. More specifically, note that the pieces X^+/Γ^+ of $S_G(K_f)$ inherit natural differentials dx_i from X^+ since it's diffeomorphic to $\mathbb{R}^n$ and so a typical element of the de Rham complex for $S_G(K_f)$ might consist of things of the form $f dx_i$ where f is a function on $S_G(K_f)$. By the very definition of $S_G(K_f)$ these start to look like functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Now, the differential $f dx_i$ only really cares about the smooth part of such a function $f = f_\infty f^\infty$ and so one quickly factors off the $(\pi^\infty)^{K_f}$ part. The remaining differential on the π_∞ part of this de Rham-like-complex part is quickly seen to be computing the $(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty)$-cohomology of $\pi_\infty \otimes V_\mathbb{C}$, and that's all she wrote.

In particular, the above indicates the following idea: automorphic representations are naturally the objects which comprise the de Rham cohomology of locally symmetric spaces. This, to me, makes their definition motivated even if one is not very interested in harmonic analysis and/or number theory.

Let us note that the multiplicity factor associated to $(\pi^\infty)K_f$ in Equation (35) is agnostic to K_f. In particular, passing to the limit over K_f (and using that π^∞ is smooth) we obtain:

Corollary 1.54. *Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is $\mathbb{Q}$-anisotropic and let $\xi : G_L \to \mathrm{GL}(V)$ be a geometrically irreducible algebraic $\overline{\mathbb{Q}}$-representation. Then, for ι_L and an embedding $L \hookrightarrow \mathbb{C}$ and all $i \geqslant 0$ there is a decomposition of smooth $G(\mathbb{A}_f)$-modules*

$$H^i(S_G, \mathscr{F}_\xi) = \bigoplus_\pi \pi^\infty \otimes H^i(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\mathbb{C})^{m(\pi)} \tag{37}$$

where π ranges over the automorphic representations of G which have central character $\chi := (\xi \mid_{A_G(\mathbb{R})^+})^{-1}$.

Remark 1.55. Now, the above is stated in the case when the locally symmetric spaces are compact. There are many ways to try and extend these results to the general situation. For example, in the general case one can try to either

(1) Consider the so-called L^2-cohomology of the Baily-Borel compactification (e.g. see [BC+83]).
(2) Consider the same cohomology groups $H^i(S_G(K_f), \mathscr{F}_\xi)$ but modify precisely what objects show up in the decomposition (e.g. see [Fra98]).

In both cases the issue is that in the case when $[G]$ is not compact (i.e. when G^{der} is not $\mathbb{Q}$-anisotropic) automorphic forms have a 'boundedness near boundary condition' which requires that either

(1) You take cohomology of the Baily-Borel compactification and extend $\mathscr{F}_\xi$ to a sheaf and take a de Rham-like cohomology that cares only about 'functions bounded near the boundary components'.
(2) You englarge your space of automorphic forms/representations.

The solution to (i) in the second list of two options is taken care of by (i) in the first list of two options.

We would like to give an example of Corollary 1.54 but to do so in a reader friendly way would probably require the reader to be very familiar with examples of compact locally symmetric spaces (e.g. those furnished as in Example 1.9). We instead opt to give an example which is not directly covered by Theorem 1.53 (because the G^{der} is $\mathbb{Q}$-anisotropic assumption doesn't hold) but is in the realm of the L^2-cohomology mentioned in Remark 1.55:

Example 1.56. Before reading this example, we suggest that the reader consults [Buz] for a discussion of the passage from classical modular forms to automorphic representations. In particular, we will use the notation φ_f to denote the automorphic form associated to f as in loc. cit. and π_f will denote the $\mathrm{GL}_2(\mathbb{A})$ orbit of f. Note that it's very important that we take the normalization $s = 2 - k$ in the discussion in [Buz, §2.3].

Take $G = \mathrm{GL}_2$. Then, as the groups $K(N)$ from are cofinal in the system of compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$ we see that S_{GL_2} is nothing other than $\varprojlim Y(N)$. Let us define for each N and for each algebraic $\overline{\mathbb{Q}}$-representation ξ of GL_2 the *parabolic* or *cuspidal* cohomology to be the space

$$H^1_!(Y(N), \mathscr{F}_\xi) := H^1(X(N), j_*\mathscr{F}_\xi) \tag{38}$$

where $j : Y(N) \hookrightarrow X(N)$ is the inclusion of $Y(N)$ into its smooth proper algebraic compactification $X(N)$. This is, in fact, an example of L^2 (or intersection cohomology).

The analogue of Corollary 1.54 is then the following decomposition for $k \geqslant 2$:

$$\varinjlim_N H^1(X(N), j_*\mathscr{F}_{\xi_k}) = \bigoplus_f \pi(f)^\infty \otimes H^1(\mathfrak{a}_{\mathrm{GL}_2}\backslash\mathfrak{gl}_2, \mathfrak{so}(2); \pi(f)_\infty \otimes V_k)^{m(\pi)} \tag{39}$$

Let us explain the terminology here:

- $\xi_k : \mathrm{GL}_2 \to \mathrm{GL}(V_k)$ is the representation $\mathrm{Sym}^{k-2}(\mathbb{Q}^2)$ (where $\mathbb{Q}^2$ is the standard representation of GL_2).
- f ranges over the cuspidal eigen-newforms in $S_k(\Gamma(N))$ as N ranges.

And, for a fixed N we get the analogue of Theorem 1.53:

$$H^1(X(N), j_*\mathscr{F}_\xi) = \bigoplus_f (\pi(f)^\infty)^{K(N)} \otimes H^1(\mathfrak{a}_{\mathrm{GL}_2}\backslash\mathfrak{gl}_2, \mathfrak{so}(2); \pi(f)_\infty \otimes V_k)^{m(\pi)} \tag{40}$$

where f travels over the eigen-newforms of level dividing N (i.e. eigenforms in $S_k(\Gamma(N))$).

Now, the Weak Multiplicity One theorem for GL_2 says that $m(\pi) = 1$ (e.g. see [HG, Theorem 11.3.4]) and the Strong Multiplicity One theorem for GL_2 (e.g. see [HG, Theorem 11.7.2] or [Sai13, Theorem 2.49.(2)]) says that $(\pi(f)^\infty)^{K(N)}$ is precisely $\mathbb{C}\varphi_f$ if f is a newform of level N. In general, Atkin-Lehner theory (e.g. see [DDT95, Theorem 1.22]) says that if f is a newform of level $N_f \mid N$ then $(\pi(f)^\infty)^{K(N)}$ is spanned by the linear independent functions φ_{f_d} for $d \mid \frac{N}{N_f}$ where $f_d(z) := f(zd)$.

We can see what is going 'by hand' in the case when $k = 2$. Namely, in this case note that $\mathscr{F}_{\xi_k}$ is nothing other than the constant sheaf $\underline{\mathbb{C}}$. Note moreover that since

$j : Y(N) \hookrightarrow X(N)$ is an embedding of a punctured curve that $j_*\underline{\mathbb{C}} = \underline{\mathbb{C}}$. Thus, we see that $H^1_!(Y(N), \mathscr{F}_{\xi_i})$ is nothing other than $H^1(X(N), \underline{\mathbb{C}})$. Then, we see from (40) that

$$H^1(X(N), \mathbb{C}) = \bigoplus_f \left(\bigoplus_{d | \frac{N}{N_f}} \mathbb{C}\varphi_{f_a} \right) \otimes H^1(\mathfrak{a}_{\mathrm{GL}_2} \backslash \mathfrak{gl}_2, \mathfrak{so}(2); \pi(f)_\infty) \tag{41}$$

where f travels over the eigennewforms of level dividing N.

One can check that this Lie algebra cohomology is 2-dimensional and thus, appealing again to Atkin-Lehner theory, we see that

$$H^1(X(N), \underline{\mathbb{C}}) \cong S_2(\Gamma(N)) \oplus S_2(\Gamma(N))) \tag{42}$$

as $\mathscr{H}_{\mathbb{C}}(\mathrm{GL}_2(\mathbb{A}_f), K(N))$-modules. But, in fact, there is another way to see this. Namely, by Hodge theory we know that we have a canonical decomposition

$$H^1(X(N), \underline{\mathbb{C}}) \cong H^0(X(N), \Omega^1_{X(N)/\mathbb{C}}) \oplus \overline{H^0(X(N), \Omega^1_{X(N)\mathbb{C}})} \tag{43}$$

Recall though that the association $f \mapsto f\,dz$ defines an isomorphism

$$S_2(\Gamma(N)) \cong H^0(X(N), \Omega^1_{X(N)\mathbb{C}}) \tag{44}$$

(e.g. see [DDT95, Lemma 1.12]). One can then transform (43) to the statement

$$H^1(X(N), \underline{\mathbb{C}}) = S_2(\Gamma(N)) \oplus \overline{S_2(\Gamma(N))} \tag{45}$$

which, with some work, can be shown to be $\mathscr{H}_{\mathbb{C}}(\mathrm{GL}_2(\mathbb{A}_f), K(N))$-equivariant. Thus, we see that Matsushima's formula in this case is nothing more than Hodge theory!

Remark 1.57. In general Matsushima's formula can be thought about as an expression of de Rhram cohomology on a locally symmetric space. If that locally symmetric space has a complex structure one can try and carry out Hodge theory in a similar way (in fact one can do this in general since locally symmetric spaces are canonically Riemannian manifolds). For example, see [BW13, Remark VII.3.5.(2)]).

Now, while we have Matushima's formula for the locally symmetric spaces for G we still feel far away from relating Galois representations to automorphic representations of G. To do this, it would be nice if we had some sort Galois action on the cohomology groups $H^i(S_G(K_f), \mathscr{F}_\xi)$ which commutes with the Hecke action. To do this the most reasonable way would be assume that

(1) The spaces $S_G(K_f)$ have the structure of a complex manifold such that the maps from Equation 18 are holomorphic.
(2) Furthermore, the complex manifolds $S_G(K_f)$ are algebraic such that the maps from Equation 18 are algebraic.
(3) The algebraic varieties $S_G(K_f)$ have models over a number field (independent of K_f) such that the maps from Equation 18 descend to these models.

in which case such a Galois action would (up to slightly modifying the coefficients) come from Grothendieck's work on étale cohomology.

Of course, this cannot happen in general as the following classical/simple observation shows:

Example 1.58. For every $n \geqslant 1$ we have that

$$\begin{aligned} \dim X_{\mathrm{GL}_n} &= \dim \mathrm{GL}_n(\mathbb{R}) - \dim \mathrm{SO}_n(\mathbb{R}) - \dim A_{\mathrm{GL}_2}(\mathbb{R})^+ \\ &= n^2 - \frac{n(n-1)}{2} = 1 \\ &= \frac{n(n+1)}{2} - 1 \end{aligned} \tag{46}$$

In particular, we see that $\dim X_{\mathrm{GL}_n}$ is odd whenever $n = 0 \mod 4$ or $n = 3 \mod 4$.

But, note that, in general, $\dim S_G(K_f) = \dim X_G$ for any G and any K_f and so, in particular, the above example shows that $\dim S_{\mathrm{GL}_n}(K_f)$ is odd whenever $n = 0 \mod 4$ or $n = 3 \mod 4$ independent of what K_f. Thus, there is no hope of fulfilling property (1) in the above in this case (let alone properties (2) and (3)).

That said, there are certainly some examples of groups G where all three of these properties can be acheived (e.g. see Example 1.30 and Example 1.19). In fact, there is a very satisfactory general theory of particular groups G for which all three of these properties can be acheived. Namely, one can think of the notion of *Shimura varieties* as being a very insightful way (due originally to Shimura and then formalized by Deligne in [Del71b]) to ensure that these three conditions are satisfied. I will not formally define Shimura variety (for this see [Del71b], [Del79], [Mil04], or [Lan17]) and instead just now assume that G is a group for which the three properties above are acheived.

Remark 1.59. Note that G which appear in the context of Shimura varieties are not the only G for which properties (1), (2) and (3) can be acheived. That said, it is a fairly broad family that is useful in many applications.

Note also that to fulfill (1), (2), and (3) in the context of Shimura varieties one needs to choose a bit more than just the choice of a G and a K_∞. Namely, one picks a certain transitive $G(\mathbb{R})$-space X for which K_∞ is then obtained as the stabilizer of the action of $G(\mathbb{R})$ on a point of x. So, the X in the theory of Shimura varieties begets the K_∞. Now, it is true that, using the language of Shimura varieties, the system of $S_G(K_f, K_\infty)$ (for the K_∞ mentioned above) is isomorphic as a system of real manifolds to $\mathsf{Sh}_{K_f}(G, X)$. But, the complex structure on $S_G(K_f, K_\infty)$ in this case depends on an identification with $\mathsf{Sh}_{K_f}(G, X)$ and, in particular, the choice of X.

This can be seen very clearly by considering the group so that $G_\mathbb{R} = U(a, b)$ where $a \neq b$. It turns out that in this situation there are essentially two choices of X both of which have underlying system $S_G(K_f)$ of smooth manifolds but for which the complex structure differs.

Let us give an example of the type of group which has a Shimura variety that will factor heavily in to the work of the author and A. Bertoloni Meli:

Example 1.60. Let E/F be a CM extension of number fields (i.e. F is totally real and E is an imaginary quadratic extension of F). Let $N \geqslant 1$ and integer and let $U_{E/F}(N)^*$ the algebraic F group defined by

$$U_{E/F}(N)^*(R) := \{g \in \mathrm{GL}_N(E^n \otimes R) : \langle gx, gy \rangle_N^* = \langle x, y \rangle_N^* \text{ for all } x, y \in R^N\} \tag{47}$$

here $\langle -,-\rangle_N^*$ is the following pairing:

$$\langle x,y\rangle_N^* := \overline{x}^\top J_N y \tag{48}$$

where

$$J_N = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \iddots & \iddots & 0 & 0 \\ 0 & \cdots & \vdots & 0 & 0 \\ (-1)^{N-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{49}$$

and $\overline{x}^\top$ denotes the conjugate transpose (where conjugation on $E^n \otimes_F R$ is inherited from the conjugation action of E (i.e. the action of the non-trivial element of $\mathrm{Gal}(E/F)$.

Let U be any inner form of $U_{E/F}(N)^*$ such that $U(F\otimes_{\mathbb{Q}}\mathbb{R})$ is not compact. Then, $G := \mathsf{Res}_{F/\mathbb{Q}} U$ is a group appearing in the theory of Shimura varieties. We will often times call the locally symmetric spaces for U *unitary Shimura varieties.*

Remark 1.61. Note that, as in Remark 1.59, to really give this example the structures mentioned in (1), (2), and (3) above we should specify the X that is in the data of a Shimura datum. Since this won't matter much here we will ignore this point.

Let us now suppose that G is a group for which (1), (2), and (3) are possible and, moreover, have been specified (i.e. an algebraic structure of a number field has been given). Let us call this number field E. Let us denote (very unconventionally) the algebraic model of $S_G(K_f)$ over E by $\mathsf{Sh}_G(K_f)$ (evoking the notion of Shimura varieties even though we aren't strictly assuming that G is contained within the framework of Shimura varieties).

Let us fix a prime number ℓ and an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. Then, note that our embedding $\iota_L : L \hookrightarrow \mathbb{C}$ then gives us (after post composing with ι) an embedding $L \hookrightarrow \overline{\mathbb{Q}_\ell}$ which we will (sloppily) also call ι_L. Note then that $\mathscr{F}_{\xi,K_f} \otimes_{\iota_L} \overline{\mathbb{Q}_\ell}$ is a $\overline{\mathbb{Q}}_\ell$-local system on $S_G(K_f)$. One can show that, in fact, this local system descends to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on E-variety $\mathsf{Sh}_G(K_f)$.

Note then that we can form the étale cohomology groups $H^i_{\text{ét}}(\mathsf{Sh}_G(K_f), \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell})$ (where for variety X over a field k and a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{F}$ on X we abbreviate the cohomology group $H^i((X_{k^{\text{sep}}})_{\text{ét}}, \mathscr{F}_{k^{\text{sep}}})$ to $H^i_{\text{ét}}(X,\mathscr{F})$). This is, thanks to the work of Grothendieck and his collaborators, a finite-dimensional $\overline{\mathbb{Q}_\ell}$-representation of Γ_E. Let us set

$$H^i_{\text{ét}}(\mathsf{Sh}_G, \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell}) := \varinjlim H^i_{\text{ét}}(\mathsf{Sh}_G(K_f), \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell}) \tag{50}$$

This space naturally has an action by $G(\mathbb{A}_f)$ which, since we assumed that our $G(\mathbb{A}_f)$ action descends to E, commutes with Γ_E and thus we get a $\Gamma_E \times G(\mathbb{A}_f)$ action.

Now, by combining the smooth base change theorem with Artin's comparison theorem we have that

$$H^i_{\text{ét}}(\mathsf{Sh}_G, \mathscr{F}_\xi \otimes_{\iota_L} \otimes \overline{\mathbb{Q}_\ell}) \cong H^i(S_G, \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell}) \tag{51}$$

which is $G(\mathbb{A}_f)$-equivariant. If we then use our isomorphism $\iota : \overline{\mathbb{C}} \cong \overline{\mathbb{Q}_\ell}$ we get an isomorphism

$$H^i_{\text{ét}}(\mathsf{Sh}_G, \mathscr{F}_\xi \otimes_{\iota_L} \otimes \overline{\mathbb{Q}_\ell}) \cong H^i(S_G, \mathscr{F}_\xi \otimes_{\iota_L} \mathbb{C}) \tag{52}$$

Now, if we assume further that G^{der} is $\mathbb{Q}$-anisotropic then by Corollary 1.54 we have a decomposition

$$H^i(S_G, \mathscr{F}_\xi \otimes_{\iota_L} \mathbb{C}) \cong \bigoplus_{\pi^\infty} \pi^\infty \otimes \sigma^i(\pi^\infty) \tag{53}$$

where we have grouped together terms indexed by π which have isomorphic finite parts π^∞. Namely, note that while this can't happen for GL_n it is possible for general G that $\pi_1 \not\cong \pi_2$ but $\pi_1^\infty \cong \pi_2^\infty$ (in which case we say that G doesn't have the *strong multiplicity one property*), and so now the sum is over all π^∞ for which there exists a π_∞ such that $\pi = \pi_\infty \otimes \pi^\infty$ occurs in Corollary 1.53. By a finiteness result of Harish-Chandra the number of π with a given isomorphism class of π^∞ is finite, and so this multiplicity term $\sigma^i(\pi^\infty)$ is finite-dimensional.

So, in summary, under our hypotheses that G^{der} is $\mathbb{Q}$-anisotropic (and our assumption G satisfies assumptions (1), (2), and (3)) we have an isomorphism of abelian groups

$$H^i_{\text{ét}}(\mathsf{Sh}_G, \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell}) \cong \bigoplus_{\pi^\infty} \pi^\infty \otimes \sigma^i(\pi^\infty) \tag{54}$$

which is $G(\mathbb{A}_f)$-equivariant and which is an isomorphism of vector spaces via the isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. In particular, since $\sigma^i(\pi^\infty)$ is the multiplicity term of an irreducible component of the $G(\mathbb{A}_f)$ action of $H^i_{\text{ét}}(\mathsf{Sh}_G, \mathscr{F}_\xi \otimes_{\iota_L} \overline{\mathbb{Q}_\ell})$ and the action of Γ_E commutes with the action of $G(\mathbb{A}_f)$ we see that Γ_E stabilizes $\sigma^i(\pi^\infty)$.

In other words, we see that using the cohomology of 'compact Shimura varieties' (or any system of locally symmetric spaces satisfying (1), (2), and (3)) we have realized our meta-principle from §1.1. Namely, we have found a vector space on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or really $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$) and $G(\mathbb{A})$ (or really $G(\mathbb{A}_f)$) act for which:

- The actions commute (by our assumptions (1), (2), and (3)).
- The $G(\mathbb{A}_f)$-module is semi-simple (using Theorem 1.53).

And so, in particular, we have figured out a way to associate to an automorphic representation π of $G(\mathbb{A})$ a Galois representation

$$\pi \rightsquigarrow \pi^\infty \rightsquigarrow \sigma^i(\pi^\infty) \tag{55}$$

which is precisely what we wanted to do!

To give an example of this in situation where G^{der} is $\mathbb{Q}$-anisotropic has the same pitfalls as mentioned before Example 1.56. But, we can do the following:

Example 1.62. One can use the decomposition in Equation 39 to obtain a decomposition

$$\bigoplus_f \pi(f)^\infty \otimes \sigma^1(\pi(f)^\infty) \tag{56}$$

Note that since GL_2 does satisfy the strong multiplicity one property the indexing set needn't change. In fact, we see that $\sigma^1(\pi(f)^\infty)$ is a 2-dimensional $\overline{\mathbb{Q}_\ell}$-representation. One can then show, in fact, that this is the Galois representation ρ_f constructed by Deligne in [Del71a].

We now summarize the above discussion:

- Automorphic representations of G are certain harmonic analytic representations of $G(\mathbb{A})$.
- They show up naturally as the de Rham cohomology of certain symmetric spaces for G (which roughly are real manifolds obtained by 'arithmetic quotients' of $G(\mathbb{R})$) as made precise (in certain cases) by Theorem 1.53.
- When G is covered by the framework of Shimura varieties (or more generally when the symmetric spaces have nice algebraic models over number fields) one can combine Matsushima's formula and Grothendieck's theory of étale cohomology to associate Galois representations to automorphic representations of G.

So, hopefully then the arrow in Equation (2) doesn't seem so far-fetched to the reader. The idea that a local Langlands correspondence should exist is quite natural if one believes in a global Langlands like correspondence. Indeed, if π is an automorphic representation of G then by Theorem 1.51 it has local constituents π_p for every prime p. Now if $\mathsf{GLC}(\pi)$ is some Galois representation associated to p this also has local constitudents $\mathsf{GLC}(\pi)_{W_{\mathbb{Q}_p}}$ for all primes p. One then might imagine that an association $\pi \mapsto \mathsf{GLC}(\pi)$ is some sort of gluing together of local correspondences $\pi_p \mapsto \mathsf{LLC}(\pi_p)$ in such a way that $\mathsf{LLC}(\pi_p) = \mathsf{GLC}(\pi)\mid_{W_{\mathbb{Q}_p}}$ (i.e. local-global compatability).

2. Part II: A taster of results and applications

2.1. **Motivation.** From Part I we know that, at least when G is sufficiently nice (i.e. is contained within the theory of Shimura varieties and satisfies the assumption that G^{der} is $\mathbb{Q}$-anisotropic) we can associate to autormorphic representations π of G Galois representations $\sigma^i(\pi_f)$. While all of the work in Part I was to have this, and we were quite excited to have seen it was doable, a lot of work is left to be done. Namely, we have the following questions in decreasing level of difficulty:

- Is $\sigma^i(\pi_f)$ directly related to the Langlands conjecture?
- Can we understand $\sigma^i(\pi_f)$ directly in terms of properties of π?
- Is $\sigma^i(\pi_f)$ even non-trivial (i.e. not a direct sum of trivial respresentations of Γ_E).

Obviously we expect this third bullet to be true since the spaces $\mathsf{Sh}_G(K_f)$ are interesting (and it's not hard to check that the third bullet has a positive answer just geometrically). The goal of Part II is to explain how people have attempted to understand this problem and how it can be used to understand the local Langlands conjecture. This part will, out of necessity, be much less rigorous. Throughout much of the following I will implicitly be assuming that G^{der} is $\mathbb{Q}$-anisotropic and simply connected.

Remark 2.1. This assumption that G^{der} is simply connected is one that consistently comes up when working with the Arthur-Selberg trace formula and ideas in its ideological orbit. The literal reason is the following funny result of Steinberg: let $\gamma \in G(\mathbb{Q})$ be semisimple, then the centralizer $Z_\gamma(G)$ is connected if G^{der} is simply connected (as an exercises, show that this fails if, for example, $G = \mathrm{PGL}_2$). This makes the theory of so-called *stable conjugacy* simpler and, in turn, makes the study of the Arthur-Selberg trace formula simpler. It also makes the formula showing up in Theorem 2.7 simpler.

2.2. **The Langlands-Kottwitz method.** The idea of Langlands, later greatly extended and clarified by Kottwitz is that one might able to understand the $\sigma^i(\pi_f)$ by point counting. More explicitly, let us make the definition

$$H^*(\mathsf{Sh}_G, \mathscr{F}_\xi) := \sum_{i=0}^{2\dim X_G} (-1)^i H^i(\mathsf{Sh}_G, \mathscr{F}_\xi) \tag{57}$$

in the Grothendieck group of $\overline{\mathbb{Q}_\ell}[G(\mathbb{A}_f) \times \Gamma_E]$-modules. In fact, using Corollary 1.54 (or more accurately the discussion surrounding Equation (54)) we can write $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$ as follows

$$H^*(\mathsf{Sh}_G, \mathscr{F}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes \sigma^*(\pi^\infty) \tag{58}$$

where $\sigma^*(\pi^\infty)$ is the virtual $\overline{\mathbb{Q}_\ell}$-representation

$$\sigma^*(\pi^\infty) = \sum_{i=0}^{2\dim X_G} (-1)^i \sigma^i(\pi^\infty) \tag{59}$$

of Γ_E.

Remark 2.2. The reason for consider this virtual representation will soon be clear, but let us remark that the reaction to its introduction is likely negative. Namely, we are interested in producing honest Galois representations, not virtual representations. One can often times surpass this issue, especially if one is interested in the local Langlands conjecture. Namely, for sufficiently nice global representations (for which a local representation can very often be embedded in as a component) and sufficiently nice ξ (which again is a situation one can usually finagle yourself in to if you're interested in local results) $\sigma^i(\pi^\infty)$ will vanish for $i \neq \dim \mathsf{Sh}_G$. For example, see the results of [VZ84]. This then shows that $\sigma^*(\pi_f)$ is nothing more than $(-1)^{\dim \mathsf{Sh}_G} \sigma^{\dim \mathsf{Sh}_G}(\pi_f)$ which is essentially just a representation.

The goal of the Langlands-Kottwitz method can now be more slightly more precisely stated as:

Question 2.3. *Can we explicitly describe the trace of the $\Gamma_E \times G(\mathbb{A}_f)$-action on $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$ in terms similar to the traces of automorphic representations?*

Note that studying the $\Gamma_E \times G(\mathbb{A}_f)$ action on $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$ is essentially equivalent to studying the $\Gamma_E \times \mathscr{H}_{\overline{\mathbb{Q}_\ell}}(G(\mathbb{A}_f))$-action and so we shall conflate the two. In particular, we will often times denote a generic element of $\Gamma_E \times \mathscr{H}_{\overline{\mathbb{Q}_\ell}}(G(\mathbb{A}_f))$ by $\tau \times f^\infty$ and discuss the trace of such an object.

To understand why this is useful to study $\sigma^*(\pi^\infty)$ we make the following observation. One can, essentially as a corollary of strong versions of the Jacobson density theorem, find for a particular π_0^∞ a function f_0^∞ such that f_0^∞ acts on $H^i(\mathsf{Sh}_G, \mathscr{F}_\xi)$, for each $i \in \{0, \ldots, 2\dim X_G\}$, as the projector to its π_0^∞ component. One can then see quite easily that for any Galois group element $\tau \in \Gamma_E$ we have that

$$\mathrm{tr}(\tau \times f_0^\infty \mid H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)) = \mathrm{tr}(\tau \mid \sigma^*(\pi_f)) \tag{60}$$

and so if we can describe the traces of the $G(\mathbb{A}_f)\times\Gamma_E$-action on $H^*(\mathsf{Sh}_G,\mathscr{F}_\xi)$ well then we can, by Equation 60, describe the trace of Γ_E on $\sigma^*(\pi_f))$ which is (ignoring the important distinction between virtual representations and true representations) enough to characterize $\sigma^*(\pi_f)$ by the Brauer-Nesbitt theorem.

OK, so now we understand why Question 2.3 is useful to understanding $\sigma^*(\pi_f)$. But what sort of answer are we looking for? Namely, how can we qualify 'similar to the traces of automorphic representations'? The answer is given by the Arthur-Selberg trace formula. This formula is famously difficult to state in any great level of generality, but in the situation we are in (where our G^{der} simply connected and $\mathbb{Q}$-anisotropic holds) things are not so bad:

Theorem 2.4. *Let χ be a quasi-character of $A_G(\mathbb{R})^+$ and let $f\in\mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f),\chi^{-1})$ (i.e. f is a smooth $\mathbb{C}$-valued function on $G(\mathbb{A}_f)$ such that $f\chi$ is compactly supported on $G(\mathbb{A})/A_G(\mathbb{R})^+$). Then,*

$$\sum_{\pi} m(\pi)\mathrm{tr}(f\mid\pi)=\sum_{\{\gamma\}} v_\gamma O_\gamma(f) \tag{61}$$

where π runs over the automorphic representations for G with central character χ.

Here $\{\gamma\}$ is running over conjugacy classes in $G(\mathbb{Q})$, v_γ is a volume term, and $O_\gamma(f)$ is an orbital integral which, surprise, is the integral of f over the orbit of γ in $G(\mathbb{A})$. For a more precise setup for these terms, see [HG].

So, a more precise version of Question 2.3 would be:

Question 2.5. *Can we explicitly describe the trace of the $G(\mathbb{A}_f)\times\Gamma_E$-action on $H^*(\mathsf{Sh}_G,\mathscr{F}_\xi)$ in terms of sums of orbital integrals?*

The answer to this question a resounding *yes* when Sh_G are certain types of PEL type Shimura varieties and we considering τ and f^∞ are of a particular type. PEL type means, roughly, Shimura varieties which can be described in terms of moduli spaces of abelian varieties with extra structure: polarizations and endomorphisms, thus the 'PE' (the 'L' stands for level as in the compact open subgroup K_f—probably the phrase 'PE Shimura varieties' would be less confusing). For GL_2 this was essentially carried out by Langlands in many cases in [Lan79] and extended to many PEL cases by Kottwitz in [Kot92b].

What is the restriction on $\tau\times f^\infty$? Well, essentially what is needed is:

- τ needs to lie in the weil group $W_{E_\mathfrak{p}}$ at some prime $\mathfrak{p}$ lying over a prime p of $\mathbb{Q}$.
- $f^\infty=f^p\mathbb{1}_{\mathcal{G}(\mathbb{Z}_p)}$ where f^p is some function on $G(\mathbb{A}_f^p)$ and $\mathbb{1}_{\mathcal{G}(\mathbb{Z}_p)}$ is the indicator function on the maximal compact subgroup $\mathcal{G}(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$ where $\mathcal{G}$ is some reductive model of G over $\mathbb{Z}_p$ (such subgroups $\mathcal{G}(\mathbb{Z}_p)$ are called *hyperspecial*—they are, in general, not unique up to conjugacy).

Why are these conditions important? Well, note that if f^p is in $\mathscr{H}_{\overline{\mathbb{Q}_\ell}}(G(\mathbb{A}_f),K^p)$ for some compact open subgroup $K^p\subseteq G(\mathbb{A}_f^p)$ (where $\mathbb{A}_f^p$ is the ring of finite adeles with trivial p-component) then the action of $\tau\times f^p\mathbb{1}_{\mathcal{G}(\mathbb{Z}_p)}$ on $H^*(\mathsf{Sh}_G,\mathscr{F}_\xi)$ can essentially be thought of as the action of $\tau\times f^p$ on $H^*(\mathsf{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p)),\mathscr{F}_\xi)$. Since τ is in $W_{E_\mathfrak{p}}$ we may as well think of this as the action of $\tau\times f^p$ on $H^*(\mathsf{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p))_{E_\mathfrak{p}},\mathscr{F}_\xi)$.

The crucial point is then this: the fact that the 'level at p' of $\mathsf{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p))$ is the hyperspecial subgroup $\mathcal{G}(\mathbb{Z}_p)$ means that the Shimura variety has good reduction. Less cryptically, there is a smooth proper 'canonical model' $\mathscr{S}_G(K^p)$ of $\mathsf{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p))$ over $\mathcal{O}_{E_\mathfrak{p}}$ and the sheaf $\mathscr{F}_\xi$ and the action of f^p also have smooth models. Since we are in the proper case we can then use smooth proper base change to relate the trace of $\tau \times f^p$ on $H^*(\mathsf{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p)), \mathscr{F}_\xi)$ to the trace of $\tau \times f^p$ on $H^*(\mathscr{S}_G(K^p)_{\overline{\mathbb{F}_p}}, \mathscr{F}_\xi)$. This is then the trace of a correspondence (see Remark 1.43) and thus can be computed using the Fujiwara-Varshavsky trace formula (e.g. see [Var07]) which is a generalization of then Grothendieck-Lefschetz trace formula to the case of correspondences which gives a 'weighted point count' of the fixed points of the correspondence.

Why are we then getting orbital integrals? Let's restrict to the case when $G = \mathrm{GL}_2$ (even though this does not satisfy our G^{der} $\mathbb{Q}$-anisotropic assumption) to get an idea. Note then that we are essentially trying to count elliptic curves over finite fields with level structure. Note that by Honda-Tate theory (see [Eis]) the isogeny classes of elliptic curves over finite fields are classified by semisimple conjugacy classes in $\mathrm{GL}_2(\mathbb{Q})$ (the unique semisimple conjugacy class which has characteristic polynomial the same as the characteristic polynomial of Frob on the Tate module of the elliptic curve). Then, counting elliptic curves isogenous to some fixed E_0 comes down to counting lattices in the adelic Tate module $\prod_\ell V_\ell(E_0)$. But such lattices are something like $\mathrm{GL}_2(\mathbb{A}_f)/\mathrm{GL}_2(\widehat{\mathbb{Z}})$ which looks like an orbit of a group action or, with a little more though, weighted orbital integral $v_\gamma O_\gamma$. So, the total count looks something like

$$\text{Total count} = \sum_{\text{isog. classes}} (\text{count within isog. class}) = \sum_{\{\gamma\}} v_\gamma O_\gamma \tag{62}$$

which is exactly what we're after.

Remark 2.6. The last part of the explanation is possibly confusingly reductive. To see what happens in a much more precise, but still reader friendly, way see [Sch11]. In particular, you don't really get a sum of orbital integrals, you get a sum of terms which are the product of a volume term, an orbital integral, a twisted orbital integral, and a trace. See Theorem 2.7 below for a better idea.

The results of Langlands-Kottwitz have been extended to an immense degree in recent years. The most general such statement is the following forthcoming result of Kisin-Shin-Zhu:

"Theorem" 2.7 ([KSZ])**.** *Assume that* Sh_G *is of abelian type. Then, there is an equality of the form*

$$\mathrm{tr}(\tau \times f^p \mathbb{1}_{\mathcal{G}(\mathbb{Z}_p)} \mid H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)) = \sum_{(\gamma_0;\gamma,\delta)} v_{\gamma_0} O_\gamma(f^p) TO_\delta(\mathbb{1}_{\mathcal{G}(\mathbb{Z}_{p^j})}) \mathrm{tr}(\xi(\gamma_0)) \tag{63}$$

Remark 2.8. The reason that this theorem is in scare quotes is that it's tremendously more complicated than what is literally written, but this gives a rough idea of the content. For example, beyond the massive technical simplifications we have made we have, in particular, greatly simplified the statement slightly than what is in [KSZ] because of our assumption that G^{der} is simply connected and anisotropic.

Here Shimura varieties of *abelian type* are a large class of Shimura varieties singled by Deligne in [Del79] and for which the vast majority of commonly encountered Shimura varieties are contained in. There is a containment of generality

$$\left\{\begin{matrix}\text{PEL type}\\ \text{Shimura varieties}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Hodge type}\\ \text{Shimura varieties}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Abelian type}\\ \text{Shimura varieties}\end{matrix}\right\} \subsetneq \{\text{Shimura varieties}\} \tag{64}$$

The Shimura varieties from Example 1.19 and Example 1.60 are abelian type but not even Hodge type.

The proof of Theorem 2.7, including the papers leading up to the actual paper [KSZ], is a tour de force of algebraic geometry, group theory, and p-adic Hodge theory. Its outline, pitifully lacking in well-deserved attributions and explanation, is as follows:

(1) Show the existence of good smooth models of Shimura varieties at p in the case of hyperspecial level at p. This was completed by Kisin in [Kis10].
(2) Show that the points of these models can be indexed into isogeny classes (a generalization of Honda-Tate theory) and can be parameterized group theoretically within a conjugacy class. This is the so-called *Langlands-Rapoport conjecture* and was finished by Kisin in [Kis17].
(3) Use the paramterization in (2) to acutally obtain a formula as in Equation (63). This was completed in [KSZ].

So, at least for abelian type Shimura varieties and in cases of good reduction, the answer to Question 2.5 has been answered.

2.3. **The Langlands-Kottwitz-Scholze method.** Let us note that the Langlands-Kottwitz method is good enough to characterize $\sigma^*(\pi)$ by Brauer-Nesbitt and the Cebotarev density theorem (ignoring the important distinction between virtual and actual representations) since the 'good reduction' hypotheses needed to apply Theorem 2.7 are true for almost all p. But, such a method is woefully insufficient to handle local questions.

Namely, an idea of how to associate a Galois representation to an admissible representation π_p of $G_p(\mathbb{Q}_p)$, where G_p is some reductive group over $\mathbb{Q}_p$, would be as folows:

(1) First realize G_p as $G_{\mathbb{Q}_p}$ for some group G over $\mathbb{Q}$ which fits into the framework for Shimura varieties.
(2) Find an automorphic representation π of G whose p^{th}-component is π_p.
(3) Get the Γ_E-representation $\sigma^*(\pi_f)$ (really a virtual representation, but we ignore this).
(4) Consider $\sigma^*(\pi_p) := \sigma^*(\pi_f)\mid_{W_{E_\mathfrak{p}}}$.

Each of these steps, if even doable, is quite difficult (most notably (2)). But, even if you could do it, you couldn't use the Langlands-Kottwitz method to get a good handle on $\sigma^*(\pi_p)$. Namely, by its very nature, the Langlands-Kottwitz method can only tell you about the natural of $\mathrm{tr}(\tau \mid \sigma^*(\pi_f))$ at places of good reduction or, in other words, when $\sigma^*(\pi_f)$ is unramified.

Again, globally this is a non-issue since representations are determined by their restrictions to the cofinitely many places they are unramified but if you are interested in working locally this forces you to deal only with unramified Galois representations and unramified representations of $G(\mathbb{Q}_p)$.

To fix this one would like a version of Theorem 2.7 where one is allowed to replace $\mathbb{1}_{\mathcal{G}(\mathbb{Z}_p)}$ with a function like $\mathbb{1}_{K_p}$ where K_p is an small compact open subgroup of $\mathcal{G}(\mathbb{Z}_p)$ (which corresponds to considering representations with arbitrarily large ramification). The realization of Scholze was that one can replace this $\mathbb{1}_{\mathcal{G}(Z_p)}$ on the left-hand side of Equation (63) with h a function of arbitrarily level $K_p \subseteq \mathcal{G}(\mathbb{Z}_p)$ if one is willing to replace that $\mathbb{1}_{\mathcal{G}(\mathbb{Z}_{p^j})}$ by a non-explicit, possibly very complicated, function.

Namely, we have the following result of Scholze:

"Theorem" 2.9 ([Sch13a]). *Assume that Sh_G is of some special types. Then, there is an equality of the form*

$$\operatorname{tr}(\tau \times f^p h) \mid H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)) = \sum_{(\gamma_0;\gamma,\delta)} v_{\gamma_0} O_\gamma(f^p) T O_\delta(\phi_{\tau,h}) \operatorname{tr}(\xi(\gamma_0)) \tag{65}$$

for some function $\phi_{\tau,h}$ and moreover this function $\phi_{\tau,h}$ depends only on data at p.

The 'special types' listed above include many cases of Shimura varieties of PEL type, for example, but is far from including all Shimura varieties of abelian type. This function $\phi_{\tau,h}$ is constructed in terms of the cohomology of deformation spaces of p-divisible groups with extra structure (a disk inside of a certain Rapoport-Zink spaces of PEL type) which depends only on $G_{\mathbb{Q}_p}$ (and several other 'at p' pieces of data). The proof of Theorem 2.9 essentially built off the work of Kottwitz in [Kot92b]

Of course, it would be nice to extend the results of Theorem 2.9 building off of the results in [KSZ] in a similar way to how [Sch13a] built off the results of [Kot92b]. This was done by the author in his thesis:

"Theorem" 2.10 ([You19]). *Assume that Sh_G is of abelian type. Then, there is an equality of the form*

$$\operatorname{tr}(\tau \times f^p h) \mid H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)) = \sum_{(\gamma_0;\gamma,\delta)} v_{\gamma_0} O_\gamma(f^p) T O_\delta(\phi_{\tau,h}) \operatorname{tr}(\xi(\gamma_0)) \tag{66}$$

for some function $\phi_{\tau,h}$ and moreover this function $\phi_{\tau,h}$ depends only on data at p.

Remark 2.11. Note that I am being incredibly sloppy in the above. Namely, the function $\phi_{\tau,h}$ depends on the choice of a reductive model $\mathcal{G}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$ and a dominant cocharacter μ of G. In particular, $\phi_{\tau,h}$ depends on $G_{\mathbb{Q}_p}$.

2.4. **The Scholze-Shin conjecture and the cohomology of unitary Shimura varieties.** While the above discussion is nice, we have yet to actually discuss in what way the representations $\sigma^*(\pi_f)$ or the cohomology $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$ has anything to do with the actual Langlands conjecture. We now remedy this.

Remark 2.12. We by no means want to imply that the following are the only results in the direction of understanding the relationship between the cohomology of Shimura varieties and the Langlands program. For instance, a conjecture about the precise relationship (at least in the case when G^{der} is simply connected and $\mathbb{Q}$-anisotropic) goes all the way back to Kottwitz's 1990 paper [Kot90] in the Ann Arbor proceedings. Great work has been done towards giving non-conjectural results in this direction since. There are countless names that could be listed in this pursuit (in no particular order): Kottwitz, Harris, Taylor, Lan, Thorne, Shin, Scholze, ... As an example to see what work has been done on unitary similitude Shimura varieties, those which have been historically the most fertile examples, one can see the 'Paris Book Project' by Harris et al. soon to have both volumes published by Cambridge University Press.

The functions $\phi_{\tau,h}$ from Theorem 2.9 (and Theorem 2.10) were begotten from a study of $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$, even though (as stated in these theorems) they are purely local objects in nature), and so if one imagines that $H^*(\mathsf{Sh}_G, \mathscr{F}_\xi)$ has anything to do with the global Langlands correspondence, one might imagine that $\phi_{\tau,h}$ has something to do with the local Langlands correspondence.

This is verified, in great form, by Scholze (in his *masters thesis!*):

"Theorem" 2.13 ([Sch13b])**.** *Let $f_{\tau,h}$ be the transfer of the function $\phi_{\tau,h}$ from Theorem 2.9 associated to the local group $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$. Then, there is a unique association $\pi_p \mapsto L(\pi_p)$ from smooth representations of $G(\mathbb{Q}_p)$ to ℓ-adic Galois representations of Γ_F such that for every function h (as in Theorem 2.9) and $\tau \in W_{\mathbb{Q}_p}$ the equality*

$$\mathrm{tr}(f_{\tau,h} \mid \pi_p) = \mathrm{tr}(h \mid \pi_p)\mathrm{tr}(\tau \mid L(\pi_p)) \tag{67}$$

holds. Moreover, this association $\pi_p \mapsto L(\pi_p)$ is the local Langlands conjecture for $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$.

Remark 2.14. Above we have apparently changed from functions $\phi_{\tau,h}$ in Theorem 2.13 to functions $f_{\tau,h}$. The relationship between them is a matter of 'base change'. In very broad terms $\phi_{\tau,h}$ is a function on $G(\mathbb{Q}_{p^j})$ for some j and $f_{\tau,h}$ is a 'matching function' on $G(\mathbb{Q}_p)$ where 'matching' means that the (twisted) orbital integrals of the former match the (stable) orbital integrals of the latter.

Remark 2.15. Note that the local Langlands conjecture for $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ was a theorem of Harris-Taylor (see [HT01]) from 2001. So, Theorem 2.13 is a reproof of the local Langlands conjecture in this case.

This theorem is quite remarkable. The local Langlands conjecture has generally been quite complicated to characterize even in the case of GL_n. Usually there are 5 properties that a bijection needs to satisfy to be the local Langlands conjecture. The fact that there is a class of geometrically defined function that controls the entire correspondence is incredible. It has also been very useful in studying local Langlands in ways that the original formulation has proven cumbersome (e.g. see [JNS17]).

The natural question one has next is:

Question 2.16. *To what extent does Theorem 2.13 hold for other groups?*

In [SS13] Scholze and S. Shin formulated a generalization of the equality given in Equation (67). Namely:

Conjecture 2.17 ([SS13])**.** *Let G be an unramified group over $\mathbb{Q}_p$ with $\mathbb{Z}_p$-model $\mathcal{G}$ and let μ be a dominant cocharacter of $G_{\overline{\mathbb{Q}_p}}$ with reflex field E. Let $\tau \in W_{\mathbb{Q}_p}$ and let $h \in C_c^\infty(\mathcal{G}(\mathbb{Z}_p))$. Let (H, s, η) be an endoscopic group for G and let h^H be the transfer of h. Then, for every tempered L-parameter φ with associated semi-simple parameter λ we have*

$$S\Theta_\varphi(f^H_{\tau,h}) = \operatorname{tr}\left(s^{-1}\tau \mid (r_{-\mu} \circ \eta \circ \lambda\,|_{W_E} \mid \cdot \mid_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\varphi(h). \tag{68}$$

Remark 2.18. In [SS13] the groups under consideration were those covered in [Sch13a] but one can easily extend this conjecture, verbatim, to the case of any group covered in [You19].

We have chosen, because it will be important momentarily, to state the conjecture rigorously. But, in this form, it's not entirely evident what the exact relationship between Equation (68) and Equation (67) are. The point is that $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ satisfies three incredibly nice properties that general groups do not:

(1) It has singleton L-packets (i.e. LLC is a bijection, not a finite-to-one map).
(2) One needn't consider endoscopic groups to characterize the Langlands correspondence.
(3) The identity map is an (irreducible) representation of GL_n.

Adjusting Equation (67) to account for these differences essentially accounts for Equation (68) in the following way:

(1) Before one needed only consider $\operatorname{tr}(f_{\tau,h} \mid \pi)$ because π is the only element of the packet of $\mathsf{LLC}(\pi)$. But, for general G we have to take a sum $\displaystyle\sum_{\pi' \in \Pi(\pi)} \operatorname{tr}(f_{\tau,h} \mid \pi)$ where $\Pi(\pi)$ is the L-packet at π. This term $S\Theta_\varphi$ in Equation (68) is, essentially, such a sum over the packet.
(2) To get an actually powerful statement one must consider endoscopic groups of G. Think about these as groups which are candidate groups for ${}^LG$ valued Galois representations to factor through. An ${}^LG$ valued Galois representation is only really pinned down well once one knows the 'minimal' endoscopic group it transfers through. So, in Conjecture 2.17 the allowance of this extra endoscopic freedom is key. If one takes the 'trivial endoscopic triple' (G, id, e) one recovers much what was in Equation (67).
(3) In general the local Langlands conjecture has ${}^LG$-valued Galois representations (roughly). These don't have traces, and so one must compose this with a representation of ${}^LG$ before taking trace. This the $r_{-\mu}$ in Equation (68). In the case of GL_n one could take standard reprsentation which is what happened in Equation (67).

We can then summarize the work of author and A. Bertoloni Meli as follows:

Theorem 2.19 (Bertoloni Meli–Y.)**.** *Conjecture 2.17 is true when G is an unramified unitary group and (H, s, η) is the trivial endoscopic triple (G, id, e).*

Remark 2.20. Note that for Conjecture 2.17 to make sense, and consequently for Theorem 2.19 to make sense, one needs to know the local Langlands conjecture for G. In this context, that of quasi-split unitary groups, this is a result of Mok in [Mok15].

We don't have enough space to explain the proof of theorem in any detail. But, let us say that one essentially follows the 4 step procedure laid out at the beginning of this subsection with special care given to choosing the group G in step (1) to have no 'global relevant endoscopy'. One then does a pseudo-stabilization technique as in [Kot92a] together with the deep results in [Shi11] and [KMSW14] to prove the result.

Remark 2.21. We remark that what is nice about Theorem 2.19 is that it is the first instance of a proven case of the Scholze–Shin conjecture in which endoscopy played a role. Namely, the previously known cases of the Scholze–Shin conjecture were for groups, roughly, of the form $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ as in [SS13] as well as version of the Scholze–Shin conjecture in the case when $G = D^\times$ as in [She19].

We also remark that the authors are currently in the process of removing the assumption that (H, s, η) is the trivial endoscopic triple.

To bring everything full circle, we note that one of the key results needed to prove Theorem 2.19 are precise descriptions of the representations $\sigma^*(\pi^\infty)$ in some situations. Namely, we have the following result:

Theorem 2.22. (Kottwitz, Bertoloni Meli–Y.) *Let G be a reductive group over $\mathbb{Q}$ which has no relevant endoscopy and for which G^{der} is $\mathbb{Q}$-anisotropic and simply connected. Suppose that Sh_G is a Shimura variety associated G with reflex field E. Then, for any irreducible algebraic $\overline{\mathbb{Q}}_\ell$-representation ξ of G and any prime $\mathfrak{p}$ of E there is a decomposition of virtual $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{A}_f) \times W_{E_\mathfrak{p}}$*

$$H^*(\mathsf{Sh}_G, \mathcal{F}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \boxtimes \sigma(\pi^\infty), \tag{69}$$

where π^∞ ranges over admissible $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{A}_f)$ such that there exists an automorphic representation π of $G(\mathbb{A})$ such that;

(1) $\pi^\infty \cong (\pi)^\infty$ (using our identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
(2) $\pi_\infty \in \Pi_\infty(\xi)$.

Moreover, for each π^∞ there exists a cofinite set $S(\pi^\infty)$ of primes p such that for each prime $\mathfrak{p}$ over E lying over p and each $\tau \in W_{E_\mathfrak{p}}$ the following equality holds:

$$\mathrm{tr}(\tau \mid \sigma(\pi^\infty)) = a(\pi^\infty)\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ \mathsf{LLC}(\pi_p))p^{\frac{1}{2}v(\tau)[E_v:\mathbb{Q}_p]\dim X_G}, \tag{70}$$

for some integer $a(\pi^\infty)$.

One should interpret this as saying that if G is some group over $\mathbb{Q}$ which satisfies the assumption that G^{der} is simply connected and $\mathbb{Q}$-anisotropic, and has 'no relevant endoscopy' then $\sigma(\pi^\infty)$ is, up to a character twist, $a(\pi^\infty)$ times a Galois representation which for almost all primes p is $\mathsf{LLC}(\pi_p)$ composed with a (specific) representation. This integer $a(\pi^\infty)$ should be thought about, roughly, as a signed version of $m(\pi)$. In fact, if ξ is 'regular' this is actually true.

Remark 2.23. The author would like to clarify the actual contribution of himself and A. Bertoloni Meli to Theorem 2.22. In particular the general idea of Theorem 2.22 is contained in [Kot92a]. There a very specific case of Theorem 2.22 is proven. Once one has singled out the notion of 'no relevant endoscopy' the rest of the proof of Theorem 2.22 is a technical exercise in the verification that the results in [Kot92a] work in this more general context using the formula given in Theorem 2.7.

If we assume further that our G is a unitary group as in Example 1.60 then we can say something even stronger:

Theorem 2.24 (Bertoloni Meli–Y.)**.** *Let $E/\mathbb{Q}$ be a CM field with F its totally real subfield. Let U be an inner form of $U_{E/F}(n)^*$ and set $G := \mathsf{Res}_{F/\mathbb{Q}}U$. Assume that G^{der} is $\mathbb{Q}$-anisotropic and has no relevant endoscopy. Let Sh_G be a Shimura variety associated to G. Then, for any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ and* any prime $\mathfrak{p}$ *of E there is a decomposition of virtual $\overline{\mathbb{Q}_\ell}[G(\mathbb{A}_f)\times\Gamma_E]$-modules*

$$H^*(\mathsf{Sh}_G,\mathcal{F}_\xi)(\chi)=\bigoplus_{\pi^\infty}\pi^\infty\boxtimes a(\pi^\infty)\left(r_{-\mu}\circ\mathsf{GLC}(\pi)\right), \tag{71}$$

where π^∞ ranges over admissible $\overline{\mathbb{Q}_\ell}$-representations of $G(\mathbb{A}_f)$ such that there exists an automorphic representation π of $G(\mathbb{A})$ such that;

(1) $\pi^\infty\cong(\pi)^\infty$ (using our identification $\overline{\mathbb{Q}}_\ell\cong\mathbb{C}$)
(2) $\pi_\infty\in\Pi_\infty(\xi)$.

and χ is some global character and $a(\pi^\infty)$ is an integer.

In fact, we are able to give an even more refined decomposition that is able to better analyze that $G(\mathbb{Q}_p)$-structure in this case. Namely:

Theorem 2.25 (Bertoloni Meli-Y)**.** *Let G be as in the previous theorem. Let π be be an automorphic representation of G such that π_∞ is discrete series. Then, for any prime $\mathfrak{p}$ of E and any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ we have a decomposition of virtual $\mathbb{Q}_\ell[G(\mathbb{Q}_p)\times W_{E_\mathfrak{p}}]$-modules*

$$H^*(\mathsf{Sh}_G,\mathcal{F}_\xi)[(\pi^\infty)^p]=\bigoplus_{\pi'_p\in\Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p),\omega_p)}\pi'_p\boxtimes\sigma((\pi^\infty)^p\otimes\pi'_p). \tag{72}$$

Remark 2.26. This theorem is plesantly surprising (at least to the author). Namely, the $G(\mathbb{A}_f)$ structure on $H^*(\mathsf{Sh}_G,\mathscr{F}_\xi)$ doesn't yield a decomposition into a $G(\mathbb{A}_f^p)\times G(\mathbb{Q}_p)$-module in a canonical, explicit way. Thus, the ability to tease out that the at-p-part of $H^*(\mathsf{Sh}_G,\mathscr{F}_\xi)$ contains precisely the information of the packet of $\sigma(\pi^\infty)\mid_{W_{E_\mathfrak{p}}}$ is unexpected. Here is where the author and Bertoloni Meli neeeded pivotally the incredibly deep work of [KMSW14] on the decomposition of the discrete spectrum for unitary groups.

Thus, we see that, at least in the case of some unitary Shimura varieties, one can explicitly describe the decomposition of the cohomology in terms of the Langlands correspondence and, moreover, that this plays a pivotal role in studying the local Langlands conjecture for unitary groups. Of course, this is only the very tip of the iceberg

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