

A SMATTERING OF REPRESENTATION THEORY

ALEX YOUICIS

1. INTRODUCTION AND MOTIVATION

Let us fix a p -adic local field F . The key theorem of Scholze’s paper [1] is Theorem 1.1 which states that if π is an admissible \mathbb{F}_p -representation and if we define the sheaf \mathcal{F}_π on $(\mathbb{P}_F^{n-1})_{\text{ét}}$ by the procedure

$$\mathcal{F}_\pi = (\pi_*^{\text{GH}} \pi)^{\text{GL}_n(F)} \quad (1)$$

where $\pi^{\text{GH}} : \mathcal{M}_\infty^{\text{LT}} \rightarrow \mathbb{P}_F^{n-1}$ is the $\Phi \times \text{GL}_n(F) \times D^\times$ -equivariant (where D is the unique central F -division algebra with invariant $\frac{1}{n}$ and Φ is the Weil descent morphism) Gross-Hopkins period map (constructed originally in [2]) then for all i and complete algebraically closed fields C containing F the $\mathbb{F}_p[W_F \times D^\times]$ -module $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$ is admissible as an $\mathbb{F}_p[D^\times]$ -module. Moreover, Scholze shows that, up to isomorphism, this procedure is independent of the choice of C . He moreover shows that the W_F -action on these cohomology groups (necessarily uniquely) extends to a continuous action of the Galois group G_F .

Thus, from this, Scholze creates a sequence of functors

$$\begin{aligned} C_{\mathbb{F}_p}^i : \{ \text{Admissible } \mathbb{F}_p[\text{GL}_n(F)]\text{-modules} \} &\rightarrow \{ \text{Admissible } \mathbb{F}_p[G_F \times D^\times]\text{-modules} \} \\ : \pi &\mapsto H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi) \end{aligned} \quad (2)$$

(where we say a $\mathbb{F}_p[G_F \times D^\times]$ -module is admissible if its underlying $\mathbb{F}_p[D^\times]$ -module is admissible) enjoying some desirable properties (e.g. $C_{\mathbb{F}_p}^i = 0$ if $i > 2(n-1)$). Moreover, Scholze then extends this to show that if (A, \mathfrak{m}) is complete Noetherian ring with finite residue field of characteristic p then there is a sequence of functors

$$C_A^i : \{ \text{Admissible } A[\text{GL}_n(F)]\text{-modules} \} \rightarrow \{ \text{Admissible } A[G_F \times D^\times]\text{-modules} \} \quad (3)$$

essentially by ‘passage to the limit’.

Remark 1.1. The definition of an admissible $A[G(F)]$ -representation, for G a connected reductive group over F , is slightly different than one might expect. For example, if $A = \mathbb{Z}_p$ then the underlying modules of such an object are torsion—they are, in fact, $\mathbb{Q}_p/\mathbb{Z}_p$ -modules. Cohomologically such objects appear by taking $\mathbb{Q}_p/\mathbb{Z}_p$ -étale cohomology, which is related to the usual cohomology groups with \mathbb{Z}_p and \mathbb{Q}_p by the usual sequence

$$0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0$$

and is related to completed cohomology (e.g. cf [3]).

If we are now in a situation where we have a hypothetical local Langlands correspondence for $\mathrm{GL}_n(F)$ (e.g. when $n = 2$ and $F = \mathbb{Q}_p$) we have a map

$$\mathrm{LL}_A : \{\text{Admissible } A[\mathrm{GL}_n(F)\text{-modules}]\} \rightarrow \{\text{Rank } n \text{ continuous } A[G_F]\text{-modules}\} \quad (4)$$

and thus obtain a functor

$$\begin{aligned} J_A : \{\text{Admissible } A[\mathrm{GL}_n(F)\text{-modules}]\} &\rightarrow \{\text{Admissible } A[D^\times]\text{-modules}\} \\ : \pi &\mapsto C_A^{n-1}(\pi)[\mathrm{LL}(\pi)] \end{aligned} \quad (5)$$

One might hope that one can show that this functor is a good candidate for the ‘Jacquet-Langlands correspondence with A -coefficients’ (something in the vein of a ‘ p -adic Jacquet-Langlands correspondence’). Such a hope has been given great evidence in the situation where it makes sense (i.e. when $n = 2$ and $F = \mathbb{Q}_p$) thanks to work of Knight of Chojecki (see [4]).

Given the above, it seems clear that having a good understanding of representation theory of groups like $\mathrm{GL}_n(F)$ is pivotal. Such a study for characteristic 0 coefficients is classical (at least if the field is also algebraically closed). But the above discussion necessitates the discussion of such representations with coefficients in characteristic p fields, in particular finite fields (so-called *modular representations*). This subject is much more difficult and less-developed.

The goal of this note is to discuss this representation theory and, in particular, compare and contrast the situations in characteristic 0 and characteristic p .

2. ADMISSIBLE REPRESENTATIONS AND HECKE ALGEBRAS

Let us begin by fixing F to be a local field of residue characteristic p , and letting E be any field. We furthermore fix a connected reductive group G over F .

We first recall the basic definition of a smooth (and admissible) E -representation of a locally compact and totally disconnected Hausdorff group H (e.g. $H = G(F)$)

Definition 2.1. *Let H be a locally compact totally disconnected Hausdorff topological group and let V be an E -space (possibly of infinite dimension). We then call a representation $\pi : H \rightarrow \mathrm{GL}_E(V)$ smooth if the action map $V \times H \rightarrow V$ is continuous when V is given the discrete topology (equivalently $V = \bigcup_K V^K$ where K ranges over the compact open subgroups of H).*

A smooth representation π is furthermore called admissible if for all compact open subgroups K of H we have that V^K is finite-dimensional as an E -space. We denote the category of smooth representations of $G(F)$ over E by $\mathrm{Rep}^\infty(H, E)$ (or just $\mathrm{Rep}^\infty(H)$ when E is clear from context).

Remark 2.2. We will often times call a smooth (resp. admissible) representation $\pi : H \rightarrow \mathrm{GL}_E(V)$ a smooth (resp. admissible) $E[H]$ -module. We will also often notationally conflate V and π .

As in the case of representations of finite groups, one is tempted to try and understand smooth representations of $G(F)$ in terms of some sort of ‘group algebra’ of $G(F)$ with coefficients in E . In a more abstract setting, we’re really asking whether there is some

‘natural ring R ’ for which we can think of admissible $E[G(F)]$ -modules in terms of left R -modules.

There is, in fact, an abstract formalism to discuss such a problem which we now recall. We begin with a definition:

Definition 2.3. *Let \mathcal{A} be an abelian category. We say that \mathcal{A} is Grothendieck if it satisfies the following conditions:*

- (1) *The category \mathcal{A} has arbitrary coproducts.*
- (2) *Filtered colimits in \mathcal{A} preserve exactness.*
- (3) *The category \mathcal{A} has a generator G (i.e. $\text{Hom}(G, -) : \mathcal{A} \rightarrow \text{Set}$ is faithful or, equivalently, every object of \mathcal{A} admits an epimorphism from $G^{\oplus I}$ for some set I)*

The theorem of Gabriel-Popescu is then the following:

Theorem 2.4 (Gabriel-Popescu embedding theorem). *Let \mathcal{A} be a Grothendieck category and let G be a generator for \mathcal{A} . Set $R := \text{End}(G)^{\text{op}}$. Then, the functor*

$$\text{Hom}(G, -) : \mathcal{A} \rightarrow R\text{-Mod}$$

is, additive, fully faithful and left exact. Moreover, has a left adjoint denoted $- \otimes_R G$.

Proof. See [5] for a short proof. □

This theorem is implicitly implied in many situations well-known to the reader:

Example 2.5. Let H be a finite group. Then, we claim that $\mathbb{C}[H]$, the left regular representation of H , is a generator for the category of finite-dimensional \mathbb{C} -representations of H . Indeed, note that since every irreducible \mathbb{C} -representation of H is a subrepresentation of $\mathbb{C}[H]$, and the category of \mathbb{C} -representations is semisimple, evidently every representation V is a quotient of a power of $\mathbb{C}[H]$. Then, it’s clear that the category of \mathbb{C} -representations of H is an abelian category.

By the Gabriel-Popescu theorem we get a fully faithful embedding of this category of representations into $\text{End}(\mathbb{C}[H])^{\text{op}}\text{-Mod}$ taking V to $\text{Hom}(\mathbb{C}[H], V) \cong V$. Note though that $\text{End}(\mathbb{C}[H])^{\text{op}} \cong \mathbb{C}[H]$ and we recover the usual statement that \mathbb{C} -representations of H fully faithful embed (in fact are equivalent to) finite length $\mathbb{C}[H]$ -modules.

Example 2.6. For finite connected p -group schemes over a perfect field k of characteristic p one gets let $\hat{\mathcal{C}}$ denote the ind-category of \mathcal{C} . One can then show $\hat{\mathcal{C}}$ has an ind-generator (with the obvious meaning) given by $\mathcal{L} := \varinjlim \ker(F^n | \mathbb{W}_n)$ (where \mathbb{W}_n is the finite length Witt scheme). Then, by a suitable (easy) generalization of the Gabriel-Popescu theorem we get a fully-faithful embedding into discrete $\text{End}(\mathcal{L})$ -modules. But, $\text{End}(\mathcal{L})$ can be identified with $W(k)[[F, V]]$ —the completed Dieudonne ring over k . In this way we recover Dieudonne theory for finite connected p -group schemes since discrete $W(k)[[F, V]]$ -modules are just finite length $W(k)[F, V]$ -modules.

The reason that this is relevant for us is the following basic observation:

Proposition 2.7. *The category $\text{Rep}^\infty(G(F))$ is Grothendieck.*

Before we begin the proof, it’s useful to construct a very useful class of representations smooth $E[G(F)]$ -modules from representations of compact open subgroups K . Namely:

Definition 2.8. Let H be a closed open subgroup of $G(F)$ and let $\sigma : H \rightarrow \mathrm{GL}_E(V)$ be a smooth representation. Then, the smooth induction $\mathrm{Ind}_H^{G(F)}\sigma$ is the set

$$\left\{ f : G(F) \rightarrow V : \begin{array}{ll} (1) & f(hg) = \sigma(h)f(g) \text{ for all } h \in H, g \in G(F) \\ (2) & \text{There exists some compact open subgroup} \\ & K_f \text{ such that } f(gk) = f(g) \text{ for all } k \in K_f \end{array} \right\}$$

We define the compact induction $\mathrm{ind}_H^{G(F)}\sigma$ to be the subset of $\mathrm{Ind}_H^{G(F)}\sigma$ consisting of those f such that $H \setminus \mathrm{supp}(f)$ is compact.

Right translation by $G(F)$ makes both $\mathrm{Ind}_H^{G(F)}\sigma$ and $\mathrm{ind}_H^{G(F)}\sigma$ smooth representations of $G(F)$. Moreover, if $G(F)/H$ is compact, then $\mathrm{ind}_H^{G(F)}\sigma = \mathrm{Ind}_H^{G(F)}\sigma$ and this $E[G(F)]$ -module is admissible if σ is an admissible $E[H]$ -module.

We then proceed with the proof of Proposition 2.7 as follows:

Proof. (Proposition 2.7) The proof of (1) and (2) in the definition of a Grothendieck category are fairly simple. The only nebulous part is the existence of a generator of $\mathrm{Rep}^\infty(G(F))$. To see this, we claim that $G := \bigoplus_K \mathrm{ind}_K^{G(F)}1$ (where 1 is the trivial representation of K) is a generator where K travels over the compact open subgroups of $G(F)$. To see this we need to show that $\mathrm{Hom}(G, -)$ is faithful. But, since

$$\mathrm{Hom}(G, V) = \prod_K V^K$$

and $V = \bigcup_K V^K$ the conclusion is clear. \square

While this theorem is nice, it does say that we can think of $\mathrm{Rep}^\infty(G(F))$ as a subcategory of a category of modules, over some ring. The ring involved however is a priori quite messy: it's

$$\mathrm{End} \left(\bigoplus_{K \subseteq G(F)} \mathrm{ind}_K^{G(F)}1 \right)^{\mathrm{op}}$$

There are two ways to get analogues of this theorem that are much more manageable in practice (in particular, which don't require us to have to worry about all compact open subgroups of $G(F)$).

The first edulcoration is of a purely formal nature. Namely, for a fixed compact open subgroup K of $G(F)$ we define a subcategory of $\mathrm{Rep}^\infty(G(F))$, denoted $\mathrm{Rep}^\infty(G(F), K)$, to be the subcategory consisting of smooth representations V of $G(F)$ such that V^K generates V as a representation. In particular, note that on such a subcategory the single representation $\mathrm{ind}_K^{G(F)}1$ becomes a generator and running through the same ideas above allows one to show (using the Gabriel-Popescu theorem) to embed $\mathrm{Rep}^\infty(G(F), K)$ into a category of modules.

To make this simpler to write, let us make the following definition:

Definition 2.9. *Let K be a compact open subgroup of $G(F)$. Then, we define the K -Hecke algebra of $G(F)$ relative to K , denoted $\mathcal{H}(G(F), K, E)$ (or just $\mathcal{H}(G(F), K)$ when E is clear from context) to be $\text{End}(\text{ind}_K^{G(F)} 1)$.*

What we then obtain is the following:

Proposition 2.10. *Let K be a compact open subgroup of $G(F)$. Then, the functor*

$$\text{Rep}^\infty(G(F), K) \rightarrow \mathcal{H}(G(F), K)\text{-Mod} : V \mapsto V^K$$

is fully faithful, left exact, and has exact left adjoint given by

$$M \mapsto M \otimes_{\mathcal{H}(G(F), K)} \text{ind}_K^{G(F)} 1$$

We will return later to the essential surjectivity of the above fully faithful functor, but for now we would like to discuss the second educloration of the above ideas. For this we make the following crucial assumption: E has characteristic 0.

The reason that E being characteristic 0 is helpful is the following result:

Proposition 2.11. *There exists an E -valued Haar measure $\mu(-) = \int_- 1 dg$ on $G(F)$.*

Remark 2.12. This proposition needs some level of explanation. Namely, Haar measures are, by definition, functions $\mu : \mathcal{B} \rightarrow [0, \infty]$ where \mathcal{B} is the Borel algebra of $G(F)$. How then does it make sense to define a ‘Haar measure valued in E ’? Given a Haar measure μ one is then able to make sense of the integration $\int_{G(F)} f d\mu$ for a function $f : (F) \rightarrow \mathbb{C}$.

Note though that if f is locally constant and compactly supported then such an integral reduces to nothing but a sum of terms of the form $f(x)\mu(K)$ for compact open subgroups K of $G(F)$. In particular, such an expression would make sense for a locally constant function $f : G(F) \rightarrow E$ assuming that one naturally can interpret $\mu(K)$ as an element of E for any compact open subgroup K of $G(F)$.

Note though that if we normalize μ so that $\mu(K_0) = 1$ for some compact open subgroup K_0 of $G(F)$ then, in fact, $\mu(K)$ is in \mathbb{Q} for any other compact open subgroup K . In fact,

$$\mu(K) = \frac{[K : K \cap K_0]}{[K_0 : K \cap K_0]}$$

and since \mathbb{Q} uniquely embeds into E we can, in fact, make sense of an expression $\int_{G(F)} f d\mu$ for $f : G(F) \rightarrow E$ locally constant compactly supported—and this is all we will really need below, and is what we mean by “there exists an E -valued Haar measure”.

If one works slightly harder, one can see that to make sense of the above one really only needs that the characteristic ℓ of E differs from that of p . Indeed, the expressions $\frac{[K : K \cap K_0]}{[K_0 : K \cap K_0]}$ can be made sense of in E as long as as there is a neighborhood basis of open subgroups of $G(F)$ for which the index of one in the other is prime-to- ℓ . If $\ell \neq p$ then one can do this precisely because $G(F)$ is locally pro- p . If $\ell = p$ one clearly sees that we are doomed to create an ‘ E -valued Haar measure’.

Using this we can define Hecke algebra like objects which are much more concrete in nature. Namely, let us make the following definition:

Definition 2.13. *The analytic Hecke algebra of $G(F)$, denoted $\mathcal{H}^{\text{an}}(G(F), E)$ (or just $\mathcal{H}(G(F))$ when E is clear from context), is the algebra of all locally constant compactly supported functions $f : G(F) \rightarrow E$ with the convolution product:*

$$(f_1 * f_2)(x) := \int_{G(F)} f_1(g) f_2(g^{-1}x) dg$$

It's not hard to show that $\mathcal{H}^{\text{an}}(G(F))$ is a non-unital E -algebra. We define a left $\mathcal{H}^{\text{an}}(G(F))$ -module M to be *smooth* if $\mathcal{H}^{\text{an}}(G(F))M = M$ (i.e. every $m \in M$ is of the form fm' for some $f \in \mathcal{H}^{\text{an}}(G(F))$ and some $m' \in M$). We assume that all $\mathcal{H}^{\text{an}}(G(F))$ -modules we consider are smooth, and so we shall sloppily denote by $\mathcal{H}^{\text{an}}(G(F))\text{-Mod}$ the category of smooth $\mathcal{H}^{\text{an}}(G(F))$ -modules.

We construct a functor $F : \text{Rep}^{\infty}(G(F)) \rightarrow \mathcal{H}^{\text{an}}(G(F))\text{-Mod}$ by declaring that for a smooth representation $\pi : G(F) \rightarrow \text{GL}_E(V)$ we define the multiplication

$$f \cdot v := \int_{G(F)} f(g) \pi(g) v dg \quad (6)$$

for $f \in \mathcal{H}^{\text{an}}(G(F))$ and $v \in V$. We denote $\pi(f)$ this resulting endomorphism of V .

One can quickly check that if we set

$$e_K := \frac{1}{\mu(K)} \mathbb{1}_K$$

for a compact open subgroup K of $G(F)$ then the operator $\pi(e_K)$ is the projection $V \rightarrow V^K$ which clearly then implies, since V is smooth, that this $\mathcal{H}^{\text{an}}(G(F))$ -module is smooth as well.

The key result is then the following:

Proposition 2.14. *The functor F is an additive equivalence of categories*

$$\text{Rep}^{\infty}(G(F)) \rightarrow \mathcal{H}^{\text{an}}(G(F))\text{-Mod} \quad (7)$$

Proof. See [6, Proposition 1, Proposition 2 §4.2]. \square

Note that $\mathcal{H}^{\text{an}}(G(F))$ is a much more concrete (non-unital) ring than the (unital) ring $\text{End} \left(\bigoplus_K \text{ind}_K^{G(F)} 1 \right)$, and so is very useful to concretely study smooth $E[G(F)]$ -modules.

One might wonder though, what the relationship between these two rings are. Of course they can't be isomorphic since one is non-unital, and the other is unital. That said, $\mathcal{H}^{\text{an}}(G(F))$ is 'made up of' unital subrings which one can then directly question if they are related to the above Gabriel-Popescu theory.

Namely, let us make the following definition:

Definition 2.15. *Let K be a compact open subgroup of $G(F)$. We define the analytic K -Hecke algebra, denoted $\mathcal{H}^{\text{an}}(G(F), K, E)$ (or just $\mathcal{H}^{\text{an}}(G(F), K)$ when E is clear from context) to be the E -subalgebra of $\mathcal{H}^{\text{an}}(G(F))$ consisting of bi- K -invariant functions $G(F) \rightarrow E$.*

Remark 2.16. It's useful to note for later that $\mathcal{H}^{\text{an}}(G(F), K)$ evidently has an E -basis consisting of the indicator functions $\mathbb{1}_{KgK}$ as g ranges over a section of $G \rightarrow K \backslash G / K$.

Note that $\mathcal{H}^{\text{an}}(G(F), K)$ is a unital E -subalgebra of $\mathcal{H}^{\text{an}}(G(F))$ since e_K acts as an identity element. In fact, it's not hard to see that the following equalities hold

$$\mathcal{H}^{\text{an}}(G(F), K) = e_K \mathcal{H}^{\text{an}}(G(F)) e_K, \quad \mathcal{H}^{\text{an}}(G(F)) = \varinjlim_K \mathcal{H}^{\text{an}}(G(F), K)$$

so that $\mathcal{H}^{\text{an}}(G(F))$ is a so-called 'idempotent E -algebra'.

It is not at all hard to show that Proposition 2.14 implies the following:

Proposition 2.17. *Let K be a compact open subgroup of $G(F)$. Then, the functor $V \mapsto V^K$ is an additive equivalence of categories*

$$\text{Rep}^\infty(G(F), K) \rightarrow \mathcal{H}^{\text{an}}(G(F), K)\text{-Mod} \tag{8}$$

Comparing this proposition to Proposition 2.10 leads one to question whether the E -algebras $\mathcal{H}(G(F), K)$ and $\mathcal{H}^{\text{an}}(G(F), K)$ are related. In particular, since they are both unital E -algebras it now makes sense to ask whether they are isomorphic. The answer is yes:

Proposition 2.18. *There is a natural isomorphism $\mathcal{H}(G(F), K) \rightarrow \mathcal{H}^{\text{an}}(G(F), K)$ for which the diagram*

$$\begin{array}{ccc} \text{Rep}^\infty(G(F), K) & \longrightarrow & \mathcal{H}(G(F), K)\text{-Mod} \\ & \searrow & \downarrow \\ & & \mathcal{H}^{\text{an}}(G(F), K)\text{-Mod} \end{array} \tag{9}$$

is 1-commutative.

Proof. See the proposition on [7, Page 59]. □

3. KAZHDAN'S ISOMORPHISM

In the introduction we mentioned a result of Kazhdan which allows one to compare smooth representations between different groups, groups like $G(F)$ and $G(F')$ where F and F' are local fields with different characteristics but the same residue characteristic. From the last section it's clear that we can do this (in certain situations) by comparing Hecke algebras for $G(F)$ and $G(F')$ which is an actually sensible question since both such objects are just E -algebras. Kazhdan achieves such a comparison in certain situations under some strong hypotheses, perhaps the most pressing for us is the assumption that E has characteristic 0. The reasons for this are multiple, but the most immediate being that Kazhdan actually compares not 'regular Hecke algebras' but 'analytic Hecke algebras'.

Let us now begin the setup necessary to state Kazhdan's theorem. Namely, let us now fix G to be a reductive group scheme over \mathbb{Z} . Recall that this means nothing more than G is a smooth affine group scheme over \mathbb{Z} such that for every $x \in \text{Spec}(\mathbb{Z})$ we have that $G_x \rightarrow \text{Spec}(k(x))$ is a connected reductive algebraic group (in the classical sense of linear algebraic groups over a field). We assume that G contains a maximal torus T (i.e. T is a smooth closed subgroup scheme of G which is étale (equiv. fppf) locally

isomorphic to a power $\mathbb{G}_{m,\mathbb{Z}}$ and for all $x \in \text{Spec}(\mathbb{Z})$ we have that $T_x \rightarrow \text{Spec}(k(x))$ is a maximal torus in G_x .

This torus T is necessarily split (i.e. isomorphic to $\mathbb{G}_{m,\mathbb{Z}}^n$) since $\pi_1^{\text{ét}}(\text{Spec}(\mathbb{Z})) = 0$ (cf. [8, Corollary B.3.6]). In particular, G is a Chevalley group.

Remark 3.1. There exist non-split reductive groups over $\text{Spec}(\mathbb{Z})$ which, by the above discussion, means that they don't admit a maximal torus (note that every group admits a torus étale locally on the base—see [8, Corollary 3.2.7]). For example see the results of [9].

In particular, we note the following useful fact:

Lemma 3.2. *For any field K the natural map*

$$\text{Hom}(\mathbb{G}_{m,\mathbb{Z}}, T) \rightarrow \text{Hom}(\mathbb{G}_{m,K}, T_K) \quad (10)$$

is an isomorphism.

Proof. This is obvious. Indeed, the claim reduces to the claim that the map

$$\text{Hom}(\mathbb{G}_{m,\mathbb{Z}}, \mathbb{G}_{m,\mathbb{Z}}) \rightarrow \text{Hom}(\mathbb{G}_{m,K}, \mathbb{G}_{m,K})$$

is an isomorphism. But one explicitly computes that the first group is nothing more than \mathbb{Z} with n corresponding to the Hopf algebra map $\mathbb{Z}[x, x^{-1}] \rightarrow \mathbb{Z}[x, x^{-1}]$ sending $x \mapsto x^n$. The same holds true with \mathbb{Z} replaced by K , and the conclusion readily follows. \square

Using this we see that for any two fields K and K' we can naturally identify the cocharacter lattices of T_K and $T_{K'}$. Let us denote by $X_*(T)$ this common abelian group. We would also like to say that we can choose a notion of dominance of cocharacters that is also independent of the field K .

To make sense of this, we first make the following observation:

Proposition 3.3. *The group G contains a Borel subgroup B (i.e. a smooth closed subgroup scheme such that B_x is a Borel in G_x for all $x \in \text{Spec}(\mathbb{Z})$).*

Proof. Let us note that $G_{\mathbb{Q}}$ contains a Borel subgroup since it's split. This is a classic result, but since I don't know a canonical reference let us quickly give an idea of a proof.

Let $\lambda \in X_*(T_{\mathbb{Q}})$ be such that for all roots $\alpha \in \Phi(G_{\mathbb{Q}}, T_{\mathbb{Q}})$ we have that $\langle \lambda, \alpha \rangle \neq 0$. This is always possible since, essentially, the set of λ such that $\langle \lambda, \alpha \rangle = 0$ for one α is a \mathbb{Z} -hyperplane in $X_*(T_{\mathbb{Q}})$, and a union of finitely many such \mathbb{Z} -hyperplane cannot equal all of $X_*(T)$. Such a λ is called *regular*. Let $Z_{G_{\mathbb{Q}}}(\lambda)$ be the scheme-theoretic centralizer of λ . We claim that $Z_{G_{\mathbb{Q}}}(\lambda) = T_{\mathbb{Q}}$.

Well, $Z_{G_{\mathbb{Q}}}(\lambda)$ is smooth and connected (see [8, Theorem 4.1.7]) and since we have an obvious inclusion $T_{\mathbb{Q}} \subseteq Z_{G_{\mathbb{Q}}}(\lambda)$ it suffices to show that we have an equality of Lie algebras $\text{Lie}(T_{\mathbb{Q}}) = \text{Lie}(Z_{G_{\mathbb{Q}}}(\lambda))$. But, note that this latter Lie algebra can be described in terms of the subalgebra of $\text{Lie}(G_{\mathbb{Q}})$ where λ acts trivially. Note though that for each root space \mathfrak{g}_{α} we have that λ acts on \mathfrak{g}_{α} by $t^{\langle \lambda, \alpha \rangle}$. By assumption this is non-trivial for all α and thus the claim follows.

Note then that since $Z_{G_{\mathbb{Q}}}(\lambda) = T_{\mathbb{Q}}$, and thus solvable, we have that $P_{G_{\mathbb{Q}}}(\lambda)$ is solvable since $P_{G_{\mathbb{Q}}} = Z_{G_{\mathbb{Q}}}(\lambda) \rtimes U_{G_{\mathbb{Q}}}(\lambda)$ (See loc. cit. for the definitions of these objects and this decomposition). But, we know that $P_{G_{\mathbb{Q}}}(\lambda)$ is a connected parabolic subgroup of G . Thus, $P_{G_{\mathbb{Q}}}(\lambda)$ being a solvable parabolic subgroup is necessarily a Borel.

To see now that G contains a Borel let $\underline{\text{Bor}}_{G/\text{Spec}(\mathbb{Z})}$ be the scheme of Borel subgroups (see [8, Theorem 5.2.11]). This is a smooth proper \mathbb{Z} -scheme (see loc. cit.) and thus, in particular, properness shows that $\underline{\text{Bor}}_{G/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Q})) = \underline{\text{Bor}}_{G/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Z}))$. We have just justified why $\underline{\text{Bor}}_{G/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Q}))$ is non-empty, from where the conclusion follows. \square

From this we see that under our identification of $X_*(T_K) \cong X_*(T_{K'})$ for any two fields K and K' we have that the B_K -dominant cocharacters are carried bijectively to the $B_{K'}$ -dominant cocharacters. Thus, using our terminology of $X_*(T)$ for the constant abelian group underlying $X_*(T_K)$ for any field K we also get a subset $X_*(T)_+ \subseteq X_*(T)$ of B_K -dominant cocharacters that is constant across all field K .

Let us now assume that F is a local field of residue characteristic p . Let us set \mathcal{O} to be the integer ring of F and π to be a uniformizer of F . We make the following definition:

Definition 3.4. *Let $\ell \geq 0$ be an integer. The ℓ^{th} congruence subgroup of $G(F)$, denoted $K_\ell(F)$, is defined as follows:*

$$K_\ell(F) := \ker(G(\mathcal{O}) \rightarrow G(\mathcal{O}/\pi^\ell))$$

Note that since $G_{\mathcal{O}} \rightarrow \text{Spec}(\mathcal{O})$ is smooth and $\text{Spec}(\mathcal{O}) \rightarrow \text{Spec}(\mathcal{O}/\pi^\ell)$ is a pro-infinitesimal thickening that the infinitesimal lifting criterion implies that the map $G(\mathcal{O}) \rightarrow G(\mathcal{O}/\pi^\ell)$ is surjective (this is just Hensel's lemma in this context). In particular, we see that we can naturally identify $G(\mathcal{O})/K_\ell(F)$ with $G(\mathcal{O}/\pi^\ell)$.

Let us fix (for this section) E to be a field of characteristic 0. Our goal is to try and compare the Hecke algebras $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ and $\mathcal{H}^{\text{an}}(G(F'), K_\ell(F'))$ (where in all cases we normalize the Haar measure so that $\mu(G(\mathcal{O})) = 1$) for a different local field F' of residue characteristic p , possibly of a different characteristic than F . To see why this might be possible, let us first recall the well-known Cartan decomposition for $G(F)$:

Lemma 3.5 (Cartan decomposition). *Let F and G be as above. Then, the following equality of sets holds:*

$$G(F) = \bigsqcup_{\lambda \in X_*(T)_+} G(\mathcal{O})\lambda(\pi)G(\mathcal{O})$$

Proof. A much more general version of this is shown in [10, §4]. \square

Now, almost by definition, an E -basis for $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ is in bijection with $K_\ell(F) \backslash G(F) / K_\ell(F)$ with

$$K_\ell(F)gK_\ell(F) \longleftrightarrow \frac{1}{\mu(K_\ell(F)gK_\ell(F))} \mathbb{1}_{K_\ell(F)gK_\ell(F)} =: e_{g, K_\ell(F)}$$

Let us note though that by the Cartan decomposition the sets $G(\mathcal{O})\lambda(\pi)G(\mathcal{O})$ form a $K_\ell(F)$ -stable partition of $G(F)$. Thus, we see that

$$G(F) = \bigsqcup_{\lambda \in X_*(T)_+} \bigsqcup_{S \in X_{\lambda, \ell}(F)} S$$

where $X_{\lambda, \ell}(F)$ denotes the set of $K_\ell(F)$ double cosets in $G(\mathcal{O})\lambda(\pi)G(\mathcal{O})$.

Now, let us note that the set $X_{\lambda, \ell}(F)$ has an obvious transitive action of $G(\mathcal{O}) \times G(\mathcal{O})$ (by left-right multiplication) and, moreover, it's clear that this action factors through

$G(\mathcal{O}/\pi^\ell) \times G(\mathcal{O}/\pi^\ell)$ under which it becomes a $G(\mathcal{O}/\pi^\ell) \times G(\mathcal{O}/\pi^\ell)$ -torsor. This makes it conceivable that the algebra might only depend on $G(\mathcal{O}/\pi^\ell)$, something that could certainly be compared between different local fields of residue characteristic p .

Namely, let us make the following definition:

Definition 3.6. *Let F and F' be local fields of residue characteristic p . Let π and π' be uniformizers of F and F' respectively. For an integer $m \geq 1$ we say that F and F' are m -close if there is an isomorphism of rings $\mathcal{O}/\pi^m \cong \mathcal{O}'/\pi'^m$.*

In particular, note that if F and F' are m -close and $\varphi : \mathcal{O}/\pi^m \rightarrow \mathcal{O}'/\pi'^m$ is an isomorphism, we get an induced isomorphism $\varphi : G(\mathcal{O}/\pi^\ell) \rightarrow G(\mathcal{O}'/\pi'^\ell)$ and thus an isomorphism

$$\varphi \times \varphi : G(\mathcal{O}/\pi^\ell) \times G(\mathcal{O}/\pi^\ell) \rightarrow G(\mathcal{O}'/\pi'^\ell) \times G(\mathcal{O}'/\pi'^\ell) \quad (11)$$

Let us denote by $\Gamma_{\lambda,\ell}(F)$ (for any F) the stabilizer of $K_\ell(F)\lambda(\pi)K_\ell(F)$ in the previously defined $G(\mathcal{O}/\pi^\ell) \times G(\mathcal{O}/\pi^\ell)$ -torsor $X_{\lambda,\ell}(F)$.

We then have the following elementary lemma:

Lemma 3.7. *The isomorphism (11) carries $\Gamma_{\lambda,\ell}(F)$ to $\Gamma_{\lambda,\ell}(F')$.*

Before we prove this it's useful to make some elementary observations. To begin, note that for any $m \geq 0$ every element of \mathcal{O}/π^m can be written in the form $\pi^k u$ where $0 \leq k \leq m$ and $u \in (\mathcal{O}/\pi^k)^\times$. Moreover, it's clear that this k is uniquely determined. Thus, we have a 'valuation function'

$$\text{val} : \mathcal{O}/\pi^m \rightarrow \{0, \dots, m\}$$

Note, moreover that if we have a ring isomorphism $\varphi : \mathcal{O}/\pi^m \rightarrow \mathcal{O}'/\pi'^m$ then π being a generator of the maximal ideal of \mathcal{O}/π^m is sent to a generator of the maximal ideal (π') of \mathcal{O}'/π'^m . In particular, $\varphi(\pi) = \pi' u$ for some $u \in (\mathcal{O}'/\pi'^m)^\times$. In particular, it's not hard then to see that the equality

$$\text{val}(\varphi(x)) = \text{val}(x)$$

holds for all $x \in \mathcal{O}/\pi^m$.

Proof. (Lemma 3.7) We begin by assuming that $G = \text{GL}_n$ and we have chosen T to be the diagonal torus. Let $\lambda : \mathbb{G}_{m,\mathbb{Z}} \rightarrow G$ be any cocharacter. We then claim that the isomorphism

$$\varphi \times \varphi : \text{GL}_n(\mathcal{O}/\pi^\ell) \times \text{GL}_n(\mathcal{O}/\pi^\ell) \rightarrow \text{GL}_n(\mathcal{O}'/\pi'^\ell) \times \text{GL}_n(\mathcal{O}'/\pi'^\ell) \quad (12)$$

carries $\Gamma_{\lambda,\ell}(F)$ to $\Gamma_{\lambda,\ell}(F')$. The claim that $(g_1, g_2) \in \Gamma_{\lambda,\ell}(F)$ is equivalent to the claim that there exists $(k_1, k_2) \in K_\ell(F) \times K_\ell(F)$ such that $g_1 \lambda(\pi) g_2 = k_1 \lambda(\pi) k_2$. Writing

$$\lambda(\pi) = \begin{pmatrix} \pi^{m_1} & & \\ & \ddots & \\ & & \pi^{m_n} \end{pmatrix}$$

it's easy to see that this condition can be phrased entirely in terms of the val of the entries of g and g' . Since this is preserved under (12) the conclusion follows.

For general G we note that since \mathbb{Z} is a Dedekind domain, it embeds into GL_n (e.g. see [11, 1.4.5]) and moreover (up to replacing our embedding by an inner conjugation of

it—see [8, Theorem 3.2.6]) we can assume that our embedding sends our fixed T into the diagonal torus for GL_n . It's easy then to reduce to the GL_n case. \square

From Lemma 3.7 we deduce that our chosen isomorphism φ induces a bijection

$$\varphi : X_{\lambda, \ell}(F) \rightarrow X_{\lambda, \ell'}(F)$$

which, in turn from the above discussion, induces an isomorphism of E -spaces

$$\varphi : \mathcal{H}^{\mathrm{an}}(G(F), K_{\ell}(F)) \rightarrow \mathcal{H}^{\mathrm{an}}(G(F), K_{\ell'}(F)) \quad (13)$$

Of course, there is no reason that this bijection of E -spaces (constructed via an explicit bijection between bases on both sides) should be a ring map. That said, the following result of Kazhdan says that this true up a small perturbation of field:

Theorem 3.8 (Kazhdan's isomorphism, [12]). *let $\ell \geq 0$ be an integer. Then, there exists some $m \geq \ell$ such that if F and F' are local fields with residue characteristic p and $\varphi : \mathcal{O}/\pi^m \rightarrow \mathcal{O}'/\pi'^m$ is an isomorphism then the induced E -space isomorphism (13) is a ring map.*

Example 3.9. Let $G = T$ a torus of dimension n and let $\ell = 0$ so that $K_0(F) = (\mathcal{O}_F^{\times})^n$. Then, the standard Satake isomorphism provides us with an E -algebra isomorphism

$$S : \mathcal{H}^{\mathrm{an}}(T(F), K_0(F)) \rightarrow E[X_*(T)]$$

which carries $\mathbb{1}_{K_0(F)\lambda(\pi)K_0(F)} \mapsto [\lambda]$. In particular, we see that if F and F' are any two local fields of the same residue characteristic p (i.e. they are 0-close) then our constructed isomorphism (13) takes $\mathbb{1}_{K_0(F)\lambda(\pi)K_0(F)}$ to $\mathbb{1}_{K_0(F')\lambda(\pi')K_0(F')}$. The fact that we have an obvious commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{\mathrm{an}}(T(F), K_0(F)) & \xrightarrow{S} & E[X_*(T)] \\ \varphi \downarrow & \nearrow S & \\ \mathcal{H}^{\mathrm{an}}(T(F'), K_0(F')) & & \end{array}$$

shows that the multiplicativity of φ reduces down to the multiplicativity of S which is well-known.

For G an arbitrary connected reductive group it's unclear whether the diagram

$$\begin{array}{ccc} \mathcal{H}^{\mathrm{an}}(G(F), K_0(F)) & \xrightarrow{S} & E[X_*(T)]^W \\ \varphi \downarrow & \nearrow S & \\ \mathcal{H}^{\mathrm{an}}(G(F'), K_0(F')) & & \end{array}$$

commutes (note that the Weyl group is split over \mathbb{Z} so, in the same vein as $X_*(T)_+$ this group makes sense independent of field K). It seems very likely to be the case though and might be able to be deduced from Maconald's formulas (cf. [13, Theorem 1.5.1]).

Moreover, there is the following conjecture of Kazhdan:

Conjecture 3.10. *The m in Theorem 3.8 can be taken to be ℓ .*

A pedantic remark should be made concerning Theorem 3.8. Namely, note that our discussion of the preservation of val shows that if $\varphi : \mathcal{O}/\pi^m \rightarrow \mathcal{O}'/\pi'^m$ is an isomorphism, then $\varphi((\pi^k)) = (\pi'^k)$ for any $0 \leq k \leq m$ and so, in particular, φ induces an isomorphism $\varphi : \mathcal{O}/\pi^k \rightarrow \mathcal{O}'/\pi'^k$ for all such k . Thus, while the isomorphism in the statement of Theorem 3.8 is with respect to m , the isomorphism in (13) is technically with respect to the induced isomorphism $\varphi : \mathcal{O}/\pi^\ell \rightarrow \mathcal{O}'/\pi'^\ell$.

The key point in needing to have F and F' be m -close for an m possibly larger than ℓ is that while the linear isomorphism (13) only requires F and F' to be ℓ -close, the proof of multiplicativity will require us to assume the further result that F and F' are m -close for a possibly larger m (to be made semi-explicit soon).

Before we continue on to the proof of Kazhdan's isomorphism, it's worth discussing the implication of most importance to us:

Corollary 3.11. *Let $\ell \geq 0$ be a fixed integer and E a field of characteristic 0. Let G be a reductive group over \mathbb{Z} . Then, there exists an $m \geq \ell$ such that if F and F' are local fields of the same residual characteristic p then the choice of an isomorphism $\varphi : \mathcal{O}/\pi^m \rightarrow \mathcal{O}'/\pi'^m$ and a splitting (B, T) of G induces an additive equivalence of categories*

$$\text{Rep}^\infty(G(F), K_\ell(F), E) \rightarrow \text{Rep}^\infty(G(F'), K_\ell(F'), E) \quad (14)$$

Proof. This follows immediately from Theorem 3.8 together with Proposition 2.17. \square

Let us note that under some sort of local Langlands type philosophy, such an equivalence should have some sort of Galois analogue. In fact, considering the following natural examples of m -close fields might give the reader a good guess at what this analogue should be:

Example 3.12. Let $m \geq 0$ be an integer. Let q be a power of p and let \mathbb{Q}_q denote the unramified extension of \mathbb{Q}_p of degree $\log_p(q)$. Then, the fields $F' = \mathbb{F}_q((t))$ and $F_m = \mathbb{Q}_q(p^{\frac{1}{p^m}})$ are m -close. Indeed, it's clear that $\mathcal{O}'/\pi'^m = \mathbb{F}_q[T]/(T^m)$. Note then that $\mathcal{O}_m = \mathbb{Z}_q[T]/(T^{p^m} - p)$. The uniformizer of \mathcal{O}_m is T and thus we see that

$$\mathcal{O}_m/\pi^m = \mathbb{Z}_q[T]/(T^{p^m} - p, T^m) = \mathbb{Z}_q[T]/(p, T^m) = \mathbb{F}_q[T]/(T^m)$$

From this one immediately thinks of the Fontaine-Winterberger theorem (see [14]) and Scholze's generalizations in the form of tilting perfectoid spaces (see [15]). Then, in some sense, we see that Kazhdan's theorem is some automorphic version of 'finite level tilting'. While this may seem far out, finite level version of the Fontaine-Winterberger theorem actually predate the theorem itself! Namely, the Galois theoretic analogue of the Kazhdan isomorphism is the following theorem of Deligne (which actually implies the Fontaine-Winterberger isomorphism):

Theorem 3.13. *Let F and F' be local fields of the same finite residue characteristic. If F and F' are m -close then there is an isomorphism of groups*

$$G_F/I_F^{(m)} \xrightarrow{\cong} G_{F'}/I_{F'}^{(m)} \quad (15)$$

where $I_F^{(m)}$ and $I_{F'}^{(m)}$ denote the m -th term in the ramification filtration.

Proof. This is [16, Theorem 2.8] \square

Let us now proceed with the proof of Theorem 3.8. The first key lemma is the following:

Lemma 3.14. *Let $C \subseteq X_*(T)_+$ be finite. Define*

$$G_C := \bigsqcup_{\lambda \in C} G(\mathcal{O})\lambda(\pi)G(\mathcal{O})$$

Then, the following holds:

- (1) *There exists an integer $m = m_C \geq \ell$ depending only on C such that for all $g \in G_C$ we have that $gK_n(F)g^{-1} \subseteq K_\ell(F)$.*
- (2) *Let $f_1, f_2 \in \mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ be such that $\text{supp}(f_i) \subseteq G_C$. Then, if F and F' are n -close then*

$$\varphi(f_1 * f_2) = \varphi(f_1) * \varphi(f_2)$$

Proof. (Sketch) To show one one first can show the claim for any cocharacter λ valued in the diagonal torus when $G = \text{GL}_n$ using the same ideas as in the proof of Lemma 3.7 (specifically thinking in terms of val) and the m we take only depends on the size of the coefficients of λ 's expression in terms of e_i^\vee . The reduction to GL_n is then clear as discussed in the proof of Lemma 3.7.

Let us now show why 2. holds. Note that since the isomorphism φ at level m for any $m \geq \ell$ (if F and F' are m -close) induces the isomorphism at level ℓ it's not hard to see that we can assume without loss of generality that $\ell = m$. Note that it suffices to consider $f_i = e_{g_i \lambda_i(\pi) g'_i, K_\ell(F)}$ where $\lambda_i \in C$ and $g_i, g'_i \in G(\mathcal{O}/\pi_\ell)$. Note then that we are trying to show that

$$\begin{aligned} \varphi(e_{g_1 \lambda_1(\pi) g'_1, K_\ell(F)} * e_{g_2 \lambda_2(\pi) g'_2, K_\ell(F)}) &= \varphi(e_{g_1 \lambda_1(\pi) g'_1, K_\ell(F)}) * \varphi(e_{g_2 \lambda_2(\pi) g'_2, K_\ell(F)}) \\ &= e_{\varphi(g_1) \lambda_1(\pi') \varphi(g'_1), K_\ell(F')} * e_{\varphi(g_2) \lambda_2(\pi') \varphi(g'_2), K_\ell(F')} \end{aligned} \quad (16)$$

Note that

$$(e_{g_1 \lambda_1(\pi) g'_1, K_\ell(F)} * e_{g_2 \lambda_2(\pi) g'_2, K_\ell(F)})(x) = c \mu(K_\ell(F) g_1 \lambda_1(\pi) g'_1 K_\ell(F) \cap x^{-1} K_\ell(F) g_2'^{-1} \lambda_2(\pi)^{-1} g_2^{-1} K_\ell(F)) \quad (17)$$

where $c = [G(\mathcal{O}) : K_\ell(F)]^{-2}$ is some measure term that explicit depends only on index $[G(\mathcal{O}) : K_\ell(F)]$. Note that since $h_i := g_i \lambda_i(\pi) g'_i$ are in G_C so that $h_i K_\ell(F) = K_\ell(F) h_i$ we have that

$$K_\ell g_i \lambda_i(\pi) g'_i K_\ell(F) = g_i \lambda_i(\pi) g'_i K_\ell(F)$$

it's not hard to see that

$$e_{g_1 \lambda_1(\pi) g'_1, K_\ell(F)} * e_{g_2 \lambda_2(\pi) g'_2, K_\ell(F)} = e_{g_1 \lambda_1(\pi) g'_1 g_2'^{-1} \lambda_2(\pi)^{-1} g_2^{-1}, K_\ell(F)}$$

It's then easy to show how this decomposes into terms that φ maps explicitly, and see that it agrees with the same construction on the other side. To keep track of the measure terms working out precisely, note that

$$\begin{aligned} \mu(K_\ell(F)) &= [G(\mathcal{O}) : K_\ell(F)] \\ &= |G(\mathcal{O}/\pi^\ell)| \\ &= |G(\mathcal{O}'/\pi'^\ell)| \\ &= [G(\mathcal{O}') : K_\ell(F')] \\ &= \mu(K_\ell(F')) \end{aligned} \quad (18)$$

□

This gives a clear avenue to try and prove Theorem 3.8. Namely, the above allows us to for any finite subset of $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ choose an n large enough so that φ commutes with multiplication on that finite subset of $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$. If we knew that as a noncommutative E -algebra $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ is finitely presented (i.e. admits a surjection from the non-commutative polynomial ring $E\langle t_1, \dots, t_r \rangle$ with finitely generated kernel) then to show that (13) is a ring map, we'd only have to show that for the $P_i(t_1, \dots, t_r)$ generating the kernel and the generators $h_1, \dots, h_r \in \mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ that

$$P_i(\varphi(h_1), \dots, \varphi(h_r)) = 0$$

Since φ is additive, if we knew that

$$\varphi(h_1^{d_1} * \dots * h_r^{d_r}) = \varphi(h_1)^{d_1} * \dots * \varphi(h_r)^{d_r}$$

then we could conclude that for all $d_i \in \mathbb{N}$ with $0 \leq d_1 + \dots + d_r \leq \max(\deg(P_i))$ then

$$P_i(\varphi(h_1), \dots, \varphi(h_r)) = \varphi(P(h_1, \dots, h_r)) = 0$$

But, this entails showing that φ is a ring map with respect to a finite list of elements which, as above, we are capable of doing.

Before we formalize this, into a proof we need to address the key assumption above—is $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ a finitely presented E -algebra? The answer, thanks to deep work of Bernstein is yes:

Lemma 3.15 (Bernstein). *For any compact open subgroup K of $G(F)$ the commutative E -algebra $Z^{\text{an}}(G(F), K) := Z(\mathcal{H}^{\text{an}}(G(F), K))$ is finite type. Moreover, the $Z^{\text{an}}(G(F), K)$ -modules $\mathcal{H}^{\text{an}}(G(F), K)$ is finitely generated.*

Proof. We give references that are more easily obtained than the original Bernstein reference, as well as being in English. Namely, see Proposition 1.10.2.1 and Theorem 1.10.3.1 of Roche's article in [17]. Translating the terminology a bit, we note that $\mathcal{H}^{\text{an}}(G(F), K)$ is obtained by letting $e = e_K$ in the terminology of loc. cit. To see that $\mathfrak{Z}^{\mathfrak{S}_e}$ is a finitely generated \mathbb{C} -algebra in the terminology of loc. cit. consider equation (1.10.2.2), the fact that \mathfrak{S}_e is finite, and Theorem 1.9.1.1. □

Before briefly explain the context of this result, we mention the following:

Corollary 3.16. *The E -algebra $\mathcal{H}^{\text{an}}(G(F), K)$ is finitely presented.*

Proof. Note that since $Z^{\text{an}}(G(F), K)$ is a finitely generated commutative E -algebra we have by Noetherianity a surjective E -algebra map $E[u_1, \dots, u_v] \rightarrow Z^{\text{an}}(G(F), K)$ with a finitely generated kernel $I = (P_1, \dots, P_r)$. Note then that we get a surjection $E\langle u_1, \dots, u_v \rangle \rightarrow Z^{\text{an}}(G(F), K)$ with ideal generated by the P_i (for any ordering of the monomials) and the relations $u_i u_j - u_j u_i$. In particular, we see that $Z^{\text{an}}(G(F), K)$ is a finitely presented E -algebra. As $\mathcal{H}^{\text{an}}(G(F), K)$ is finite over it it's a finitely presented $Z^{\text{an}}(G(F), K)$ -algebra. Then, the fact that finite presentation is preserved by composition shows that $\mathcal{H}^{\text{an}}(G(F), K)$ is a finitely presented E -algebra as claimed. □

The proof of Lemma 3.15 is in the context of the *Bernstein center* and *Bernstein decomposition* of the category $\text{Rep}^\infty(G(F), E)$. To fully explain this would lie outside

the scope of this note (Roche’s article in [17] is a very nice introduction) but let us suffice to say roughly the contents of these results.

Recall first the following well-known definition:

Definition 3.17. *Let L be a rational Levi subgroup of G and let P be its associated parabolic. For any smooth representation σ of $L(F)$ we denote by $\text{Ind}_L^G \sigma$ the representation $\text{Ind}_{P(F)}^G(F) \tilde{\sigma}$ where $\tilde{\sigma}$ is the inflation to $P(F)$ via the decomposition $P = R_u(P) \rtimes L$. A smooth representation π of $G(F)$ is called supercuspidal if it is not a subquotient of any representation of the form $\text{Ind}_L^G \sigma$ for a proper Levi subgroup $L \subsetneq G$.*

The first key to understanding the context of Lemma 3.15 is the following classic result:

Theorem 3.18. *Let π be an irreducible smooth (and thus necessarily admissible by Harish-Chandra’s theorem—see [18, Theorem 12]) $E[G(F)]$ -module. Then, there exists a Levi subgroup L of G and a supercuspidal representation σ of $L(F)$ such that π is a subquotient of $\text{Ind}_L^G \sigma$. Moreover, the pair (L, σ) (called the supercuspidal support of π) is unique up to $G(F)$ -conjugacy.*

Proof. See [17, Proposition 1.7.2.1, Roche]. □

This then motivates the consideration of pairs (L, σ) as being fundamental to the study of $\text{Rep}^\infty(G(F))$. In fact, if one uses a slightly coarser equivalence than just conjugacy, namely $(L, \sigma) \sim (L', \sigma')$ if L and L' are conjugate by some $g \in G(F)$ and $\sigma = \sigma' \chi$ for a homomorphism $\chi : M(F)/M^\circ \rightarrow C^\times$ (see [17, §1.4.1, Roche] for the definition of M°) one obtains a decomposition of the category $\text{Rep}^\infty(G(F))$ itself. Namely, let us denote by $\mathfrak{B}(G)$ the set $\{(L, \sigma)\} / \sim$.

Then, the remarkable theorem of Bernstein is the following:

Theorem 3.19 (Bernstein). *Let V be an object of $\text{Rep}^\infty(G(F))$. Then, V admits a unique decomposition*

$$V = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} V^{\mathfrak{s}} \tag{19}$$

where every irreducible subquotient of $V^{\mathfrak{s}}$ has supercuspidal support in \mathfrak{s} .

Applying this theorem to the regular representation of $G(F)$ on $\mathcal{H}^{\text{an}}(G(F))$ gives us subalgebras $\mathcal{H}^{\text{an}}(G(F))^{\mathfrak{s}}$ with centers we denote $\mathfrak{Z}^{\mathfrak{s}}$. Bernstein shows that these algebras $\mathfrak{Z}^{\mathfrak{s}}$ are isomorphic to the identity functor on the subcategory of objects V in $\text{Rep}^\infty(G(F))$ with $V = V^{\mathfrak{s}}$. He then uses this description to concretely describe these algebras $\mathfrak{Z}^{\mathfrak{s}}$ as the global sections of the structure sheaf on an algebraic variety over E . The claim that $Z(\mathcal{H}^{\text{an}}(G(F), K))$ is finitely generated then follows by showing it is isomorphic to a finite product of $\mathfrak{Z}^{\mathfrak{s}}$ ’s. Moreover, the finite generation of $\mathcal{H}^{\text{an}}(G(F), K)$ over its center is also done in the light of these representation theoretic notions.

With all of this setup, we can now finally prove Theorem 3.8:

Proof. (Theorem 3.8) It is not hard to see that if $S \subseteq X_*(T)_+$ is a finite set generating $X_*(T)_+$ as a monoid, and if we set

$$X_S(F) := \bigcup_{\lambda \in S} X_{\lambda, L}(F)$$

then the set e_x for x in the finite set $X_S(F)$ is a set of E -algebra generators for $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$. Let P_1, \dots, P_r denote the generators of the kernel of the E -algebra map

$$E\langle t_x \rangle \rightarrow \mathcal{H}^{\text{an}}(G(F), K_\ell(F)) : t_x \mapsto e_x$$

(there are only finitely many by Corollary 3.16). Let $d := \max \deg P_i$. Note that the union of the set of all combinations of products of the double cosets in S is compact, and so will be contained in G_C s in Lemma 3.14 for some finite subset $C \subseteq X_*(T)_+$. Take $m = m_C$. Then, Lemma 3.14 together with the discussion immediately following it imply that the φ in (13) will be a ring map if F and F' are m -close. \square

Remark 3.20. Note that, a priori, to make sense of all of the above we really only needed to assume that G was a split reductive group over $\mathbb{Z}_{(p)}$. One might think that this adds a level of generality but, in fact, it does not. Indeed, thanks to the Chevalley-Demazure theorem (cf. Theorem [8, Theorem 1.2 and Proposition 1.3]).

Let us note that we have now have, thanks to Corollary 3.11, satisfactorily made sense of how to compare admissible representations of $G(F)$ and $G(F')$ in certain situations even when F and F' have different characteristic. Of course, as stated in the introduction, to try and compare a function field analogue of Scholze's construction to the p -adic setting what we really want is something like the Kazhdan isomorphism for Hecke algebras valued in characteristic p fields. This is, from what I gather, completely open territory.

I list here some questions and comments that might be helpful towards thinking in such a direction:

- (1) Can one phrase the Kazhdan isomorphism purely in terms of $\mathcal{H}(G(F), K_\ell(F))$? Namely, from Proposition 2.18 we know that $\mathcal{H}^{\text{an}}(G(F), K_\ell(F))$ and $\mathcal{H}(G(F), K_\ell(F))$ are isomorphic. Can we rephrase the induced isomorphism

$$\mathcal{H}(G(F), K_\ell(F)) \rightarrow \mathcal{H}(G(F'), K_\ell(F'))$$

in a way that makes sense regardless of what E is?

- (2) As observed in Example 3.9 there is a connection between the Kazhdan isomorphism and the Satake isomorphism. Classically the Satake isomorphism is only stated with coefficients in a characteristic zero. That said, recent work of Herzig and Henniart-Vigneras (see [19] and [20]) gives a characteristic p analogue of the Satake isomorphism. Can this be analyzed to study a Kazhdan like isomorphism in characteristic p , at least in the hyperspecial setting?
- (3) Is it even true that $\mathcal{H}(G(F), K_\ell(F), E)$ is a finitely generated E -algebra if E is of characteristic p ? This seems to be true in some basic cases (e.g. at hyperspecial level), but I'm not sure of any general result. The work of Ollivier seems to be important here (at least at pro- p Iwahori level—see the next section).

4. THE PRO- p IWAHORI HECKE ALGEBRA

While representations valued in characteristic p fields is definitely a more complicated setting than the characteristic 0 coefficient setting (e.g. because it lacks an analytic model for the Hecke algebra) it does have one distinct advantage. Moreover, after

explaining this advantage we will also be able to give an actual concrete way in which the modular theory is ‘worse’ than the characteristic 0 setting.

Remark 4.1. When we speak of *modular representations* we mean smooth representations valued in characteristic p fields.

To explain this strange distinction between characteristic p and characteristic 0, we begin by recalling the following basic fact from algebra:

Lemma 4.2. *Let G be a finite p -group and let X be a finite set. Then,*

$$\#X = \#X^G \pmod p$$

Proof. This is immediate since we know that

$$X = \bigsqcup_{x \in G \backslash X} Gx$$

so that

$$\#X = \sum_{x \in G \backslash X} \#Gx = \sum_{x \in G \backslash X} [G : \text{Stab}(x)]$$

If $x \notin X^G$ then $p \mid [G : \text{Stab}(x)]$ and the conclusion follows. \square

From this we deduce the following:

Corollary 4.3. *Let G be a finite p -group and let $\rho : G \rightarrow \text{GL}_{\overline{\mathbb{F}_p}}(V)$ be a finite-dimensional $\overline{\mathbb{F}_p}$ -representation. Then, $V^G \neq 0$.*

Proof. It’s clear that ρ is defined over some finite subfield \mathbb{F}_q . The claim then immediately follows from the previous lemma by noting that $V^G = 0$ implies that $\#V^G = 1$. \square

We will now bootstrap this to say something fascinating about smooth $\overline{\mathbb{F}_p}$ -representations of $G(F)$ for a local field F of residue characteristic p and G/F a reductive group. Let us also assume that the coefficient field E is $\overline{\mathbb{F}_p}$ for the remainder of this section.

Proposition 4.4. *Let $\pi : G(F) \rightarrow \text{GL}(V)$ be a smooth $\overline{\mathbb{F}_p}$ -representation. Then, for any pro- p compact open subgroup K we have that $V^K \neq 0$.*

Proof. Let us restrict ρ to K . Note then that for any vector $v \in V$ we have that $\text{Stab}(v)$ is an open subgroup of K . In particular, we know that $K/\text{Stab}(v)$ is finite, and thus the $\overline{\mathbb{F}_p}[K]$ -module generated by v is finite-dimensional—let’s call it V . Note then that since $\rho : K \rightarrow \text{GL}_{\overline{\mathbb{F}_p}}(V)$ is continuous that it factors through a finite p -group quotient. The claim then follows from Corollary 4.3. \square

Let us note that for any G the group $G(F)$ contains compact open pro- p subgroups. Indeed, one need merely fix an embedding $G \hookrightarrow \text{GL}_n$ and take $G(F) \cap K_1(F)$ where $K_1(F)$ is the first congruence subgroup of $\text{GL}_n(\mathcal{O})$.

That said, there is a generally ‘optimal’ type of pro- p subgroups known as *pro- p Iwahori subgroups*:

Definition 4.5. *Let $B(G, F)$ denote the building for G over F (see [21] for a definition of the building of G). A subgroup I of $G(F)$ of the form $\text{Stab}(\mathcal{A})$ for an alcove \mathcal{A} in $B(G, F)$ is called an Iwahori subgroup of $G(F)$. There is a unique pro- p Sylow subgroup $I(1)$ of I . Such subgroups of $G(F)$ are called pro- p Iwahori subgroups of $G(F)$.*

All Iwahori subgroups of $G(F)$ are conjugate (see [21, §3.7]) since $G(F)$ acts transitively on the alcoves in $\mathcal{B}(G, F)$. In particular, there is really no ambiguity in saying ‘the’ Iwahori and ‘the’ pro- p Iwahori subgroup of $G(F)$.

If the reader is not very familiar with buildings, they can content themselves with the following example, which will be the one of most interest to us in this section:

Example 4.6. Take $G = \mathrm{GL}_n$. Then, the Iwahori subgroup of $\mathrm{GL}_n(F)$ is

$$I = \{g \in \mathrm{GL}_n(\mathcal{O}) : (g \pmod{\pi}) \in B(\mathcal{O}/\pi)\}$$

where B is the standard Borel subgroup of upper triangular matrices in G . The pro- p Iwahori subgroup of $\mathrm{GL}_n(F)$ is

$$I(1) = \{g \in \mathrm{GL}_n(\mathcal{O}) : (g \pmod{\pi}) \in U(\mathcal{O}/\pi)\}$$

where $U = R_u(B)$ is the group of unipotent upper triangular matrices in GL_n .

Remark 4.7. If the reader is curious about the above off-hand remark that pro- p Iwahori subgroups ‘optimal’ pro- p subgroups of $G(F)$, see [22, Proposition 3.3.3] to see justification, at least in the simply connected semisimple case.

Now, since the pro- p Iwahori subgroup of $G(F)$ always has fixed points for any smooth representation, we might hope that the functor $V \mapsto V^{(1)} := V^{I(1)}$ to $\mathcal{H}(G(F), I(1))$ -modules is not too lossy (e.g. it’s not at all lossy if V is irreducible). In particular, one might wonder whether or not whether the functor

$$\mathrm{Rep}^\infty(G(F)) \rightarrow \mathcal{H}(G(F), I(1))\text{-Mod} : V \mapsto V^{(1)}$$

is an equivalence of categories. Of course, this cannot literally be true since if $V^{(1)}$ does not generate V then V and the subrepresentation generated by $V^{(1)}$ will map to isomorphic objects, even though they are not isomorphic representations necessarily. But, one might still wonder whether the fact that $\mathrm{ind}_{I(1)}^{G(F)} 1$ is ‘almost a generator’ (since everything at least has non-zero $I(1)$ -invariants) allows us to say something meaningful about this functor.

Perhaps a good place to start, which remedies the above concern, is with the functor

$$\mathrm{Rep}^\infty(G(F), I(1)) \rightarrow \mathcal{H}(G(F), I(1))\text{-Mod} : V \mapsto V^{(I)}$$

which, by Proposition 2.10, is fully faithful. Is this functor an equivalence? We observed in Proposition 2.17 that this is always the case with characteristic 0 coefficients, and if there was a compact open subgroup $K \subseteq G(F)$ where one might hope Proposition 2.17 holds true, the pro- p Iwahori subgroup is certainly a good candidate.

But, this is our first major difference between the characteristic 0 and modular theory:

Theorem 4.8 (Breuil, Ollivier). *The functor*

$$\mathrm{Rep}^\infty(G(F), I(1)) \rightarrow \mathcal{H}(G(F), I(1))\text{-Mod} : V \mapsto V^{(I)}$$

is an equivalence if $F = \mathbb{Q}_p$ and is not essentially surjective if $\mathrm{char} F = p$ or if $\mathcal{O}/\pi \neq \mathbb{F}_p$.

Proof. See [23] for a full discussion. □

The examples constructed in [23] are quite complicated, and it would be fascinating to see a talk devoted entirely to this topic.

In an entirely different vein work of Schneider in [24] amazingly shows how to fix not only the issue brought to light by Theorem 4.8 but also the fact that $\text{ind}_{I^{(1)}}^{G(F)} 1$ is not a generator of $\text{Rep}^\infty(G(F))$. Namely, he shows that if one *derives* both sides accordingly that one actually recovers an actual equivalence.

Namely, let us denote by $D(G)$ the derived category of the abelian category $\text{Rep}^\infty(G(F))$. Moreover, let us fix an injective resolution $\text{ind}_{I^{(1)}}^{G(F)} 1 \rightarrow \mathcal{I}^\bullet$ in $\text{Rep}^\infty(G(F))$ and set $\mathcal{H}(G(F), I^{(1)})^\bullet$ to be the DGA obtained as $\text{End}^\bullet(\mathcal{I}^\bullet)$ (see [24] for details).

Then, the theorem of Schneider states the following:

Theorem 4.9 (Schneider). *There is an equivalence of triangulated categories*

$$D(G) \rightarrow D(\mathcal{H}(G(F), I^{(1)})^\bullet)$$

such that the composition with

$$D(\mathcal{H}(G(F), I^{(1)})^\bullet) \rightarrow D(\overline{\mathbb{F}}_p\text{-Mod})$$

is the derived functor of the $I^{(1)}$ -invariants functor.

Remark 4.10. Note that since $I^{(1)}$ is pro- p that the invariants functor is not right exact, which explains the need/desire to derive it in the latter part of the above theorem.

It is worth noting that if we want to apply these ideas to the setting of Scholze's paper and its generalizations one will also need an analogue of Kazhdan's theorem that works even when G is not split since, after all, we need to apply it to D^\times . That said, at least in the D^\times setting Badulescu has written down an appropriate analogue of the Kazhdan isomorphism (with characteristic 0 coefficients). See [25, Theorem 2.13].

Let us end this section with some questions and comments:

- (1) A priori applying a (conjectural) characteristic p version of Kazhdan's theorem is a little touchy since we no longer have an equivalence of smooth representations with Hecke modules. That said, we still know from Proposition 2.10 that for smooth representations π and ρ generated by their $I^{(1)}$ -invariants that $\pi \cong \rho$ if and only if their associated $\mathcal{H}(G(F), I^{(1)})^\bullet$ -modules are isomorphic. This could in theory relate the Lubin-Tate constructions in different characteristics (assuming a Kazhdan type theorem for $I^{(1)}$) assuming that all objects involved are generated by their pro- p Iwahori invariants.
- (2) Schneider's proof of Theorem 4.9 is purely formal—in particular, the fact that $\text{ind}_{I^{(1)}}^{G(F)} 1$ somehow becomes a generator in the derived setting is entirely nebulous to me. Can we understand this result more concretely?
- (3) Does $D(G)$ have the type of representation theoretic notions that we'd need to do much of smooth representation theory (e.g. a notion of supercuspidal support, etc.)?
- (4) Given the importance of $I^{(1)}$ representation theoretically it's natural to wonder if the $I^{(1)}$ -level of the Lubin-Tate tower hold specific significance.
- (5) Can we understand Badulescu's proof better? How does he get around the issue that there is no consistent definition of $X_*(T)$, etc.?

5. CLASSIFICATION OF MODULAR REPRESENTATIONS

The previous section may leave the reader feeling like modular representations of p -adic groups are impenetrable. This is not entirely wrong, but hard work of many people have provided some of the bare bones tools for their study (including an essentially complete theory for GL_2).

In particular, we would like to state the results of the recent work of Abe, Henniart, Herzig, and Vigneras.

For this section let F be a local field of residue characteristic p and E an arbitrary field of characteristic p .

To state it we first need to recalled the generalized Stein representation associated to a pair $Q \subseteq P$ of parabolic subgroups of G :

Definition 5.1. *Let P and Q be rational parabolic subgroups of G with $Q \subseteq P$. The generalized Stein representation associated to the pair (Q, P) is defined as follows:*

$$\mathrm{St}_Q^P := \frac{\mathrm{ind}_{Q(F)}^{P(F)} 1}{\sum_{Q' \subseteq Q \subseteq P} \mathrm{ind}_{Q'(F)}^{P(F)} 1}$$

We have the following difficult theorem concerning generalized Steinberg representations:

Theorem 5.2 (Grosse-Klone, Ly). *For any pair $Q \subseteq P$ of rational parabolics in G the generalized Stein representation St_Q^P is irreducible and admissible.*

Proof. This is the contents (specifically Theorem 3.1) of [26]. \square

As is standard in the classical theory of smooth representations over \mathbb{C} , parabolic induction plays a huge role in the study of the smooth representations of $G(F)$. But, in a slight deviation from the classical theory, in the modular world one extends representations from a parabolic to a largest parabolic containing it before inducing.

Less cryptically:

Theorem 5.3. *Let L be a rational Levi subgroup of G and let P be its associated parabolic. Given a smooth representation σ of $L(F)$ let $\tilde{\sigma}$ denote the inflation to $P(F)$. Then, there exists a largest parabolic $P(\sigma)$ containing P for which $\tilde{\sigma}$ extends to $P(\sigma)(F)$. This extension is unique, and denoted σ^e . The representation σ^e is smooth, admissible, and/or irreducible if σ is.*

Proof. This is [27, Corollary 1, II.7]. \square

Let us consider triples of the form (P, σ, Q) where P and Q are rational parabolics, σ is a smooth representation of $L(F)$ (where L is the Levi subgroup of P), and $P \subseteq Q \subseteq P(\sigma)$. Define two triples (P, σ, Q) and (P', σ', Q') to be equivalent if they are $G(F)$ -conjugate. We say that a triple (P, σ, Q) is supercuspidal if σ is an irreducible supercuspidal representation of $L(F)$.

For a triple (P, σ, Q) we define the smooth representation $I(P, \sigma, Q)$ of $G(F)$ as follows:

$$I(P, \sigma, Q) := \mathrm{Ind}_{P(\sigma)(F)}^{G(F)} (\sigma^e \otimes \mathrm{St}_Q^{P(\sigma)}) \quad (20)$$

and call it the *associated representation*.

The landmark result of Abe, Henniart, Herzig, and Vigneras is the following:

Theorem 5.4. *For a supercuspidal triple (P, σ, Q) we have that $I(P, \sigma, Q)$ is irreducible and admissible. Moreover, $I(P, \sigma, Q)$ and $I(P', \sigma', Q')$ are isomorphic if and only if (P, σ, Q) and (P', σ', Q') are equivalent. Finally, every irreducible admissible representation of $G(F)$ is isomorphic to a $I(P, \sigma, Q)$.*

Proof. In the algebraically closed setting this precisely Theorems 1, 2, and 3 of [27]. In the non-algebraically closed setting this is a result of Henniart-Vigneras and is contained in [28]. \square

This theorem is truly incredible. In particular, due to the more rigid nature of the modular setting and the notion of extended representations we see that we have a literal expression for every irreducible admissible representation of $G(F)$. Indeed, contrast this to the classical complex theory where every irreducible admissible representation of $G(F)$ has a supercuspidal support (P, σ) but $\text{Ind}_P^G \sigma$ is rarely irreducible and all we know is that π is an irreducible subquotient of $\text{Ind}_P^G \sigma$.

While this theorem does give a beautiful classification of modular representations of $G(F)$ that reduces their classification to supercuspidals, it doesn't give one an explicit classification (i.e. classify the supercuspidal pairs (P, σ, Q)).

So, to end the contentful part of this section we would like to record a much more concrete version of the above theorem for GL_2 due to Barthel-Livne, together with a classification of the supercuspidal triples thanks to Breuil in the $\text{GL}_2(\mathbb{Q}_p)$ setting (albeit phrased in different language).

Theorem 5.5 (Barthel-Livne). *Let F be a local field of residue characteristic p . Then, every irreducible admissible $\overline{\mathbb{F}}_p$ -representation of $\text{GL}_2(F)$ is contained in the following list:*

- (1) *A representation of the form $\text{Ind}_B^{\text{GL}_2}(\chi_1 \otimes \chi_2)$ where B is the standard Borel in GL_2 of upper triangular matrices and χ_i are continuous characters $\chi : F^\times \rightarrow \overline{\mathbb{F}}_p^\times$. Here $\chi_1 \otimes \chi_2$ denotes the character of the split diagonal torus T sending $\text{diag}(a, b) \in T(F)$ to $\chi_1(a)\chi_2(b)$.*
- (2) *Characters of $\text{GL}_2(F)$ of the form $\chi \circ \det$ where $\chi : F^\times \rightarrow \overline{\mathbb{F}}_p^\times$ is a continuous character.*
- (3) *The representations $\text{St}_B^{\text{GL}_2} \otimes (\det \circ \chi)$ where $\det \circ \chi$ are as in 2.*
- (4) *Irreducible admissible supercuspidal representations of $\text{GL}_2(F)$.*

Moreover, the characters χ, χ_1 , and χ_2 listed above are uniquely determined.

Proof. This is the content of [29]. \square

Thus, we see that the mystery of the irreducible admissible $\overline{\mathbb{F}}_p$ -representations of $\text{GL}_2(F)$ lies in the determination of the supercuspidal such objects. When $F = \mathbb{Q}_p$ Breuil has given a spectacular classification of these.

To do this, it's useful to first make the following definition:

Definition 5.6. *A weight is an irreducible representation $\rho : \text{GL}_2(\mathbb{F}_p) \rightarrow \text{GL}_{\overline{\mathbb{F}}_p}(V)$ where V is finite-dimensional over $\overline{\mathbb{F}}_p$.*

The reason that weights are relevant to us is by the following elementary result:

Lemma 5.7. *If $\rho : \mathrm{GL}_2(\mathbb{F}_p) \rightarrow \mathrm{GL}_{\overline{\mathbb{F}_p}}(V)$ is a weight then the inflation $\bar{\rho} : \mathrm{GL}_2(\mathbb{Z}_p) \rightarrow \mathrm{GL}_{\overline{\mathbb{F}_p}}(V)$ is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Z}_p)$. This association creates a bijection*

$$\{\text{Weights}\} / \approx \longrightarrow \{\text{Irreducible admissible } \overline{\mathbb{F}_p}[\mathrm{GL}_2(\mathbb{Z}_p)\text{-modules}\} / \approx$$

Proof. This is easy, for a proof see [30, Corollary 12] □

Now, if you take a weight V and interpret it as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$, then one can take the compact induction $\mathrm{ind}(V) := \mathrm{ind}_{\mathrm{GL}_2(\mathbb{Z}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} V$. This will not, in general, be irreducible or admissible. For example, if V is the trivial weight 1 then Frobenius reciprocity implies that

$$\mathrm{Hom}(\mathrm{ind}(1), \pi) = \pi^{\mathrm{GL}_2(\mathbb{Z}_p)}$$

so that every irreducible unramified representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is a quotient of $\mathrm{ind}(1)$.

That said, Breuil figured out precisely quotients of $\mathrm{ind}(V)$ are irreducible and admissible. To state it we need the following version of a mod p Satake isomorphism:

Lemma 5.8. *For any weight V there is a canonical and explicit isomorphism*

$$\mathrm{End}(\mathrm{ind}(V)) \cong \overline{\mathbb{F}_p}[T_1, T_2^{\pm 1}]$$

Proof. See [30, Theorem 21]. □

The remarkable theorem of Breuil is then as follows:

Theorem 5.9 (Breuil). *The irreducible admissible supercuspidal $\overline{\mathbb{F}_p}[\mathrm{GL}_2(\mathbb{Q}_p)]$ -modules are precisely those of the form $\mathrm{ind}(V) \otimes_{\mathrm{End}(\mathrm{ind}(V))} \chi'$ where $\chi' : \mathrm{End}(\mathrm{ind}(V)) \rightarrow \overline{\mathbb{F}_p}$ is an $\overline{\mathbb{F}_p}$ -algebra homomorphism such that $\chi'(T_1) = 0$.*

Proof. This was originally shown in [31] and [32]. See [30, §10] for a nice explanation of an alternative proof. □

Thus, all that remains to do is classify the weights V . This is a classical problem of modular representation theory which has a surprisingly simple solution:

Theorem 5.10. *Every weight V is isomorphic to $(\mathrm{Sym}^{a-b} \overline{\mathbb{F}_p}^2) \otimes \det^b$ for a unique pair of integers (a, b) with $0 \leq a - b \leq p - 1$ and $0 \leq b < p - 1$.*

Proof. See [33, Proposition 2.17]. □

This completely shuts the door on the classification of irreducible admissible $\overline{\mathbb{F}_p}[\mathrm{GL}_2(\mathbb{Q}_p)]$ -modules.

We end this section with some questions and comments:

- (1) Given Theorem 5.4 one has a very well-defined notion of supercuspidal support. Consider this is the first step necessary to perform Bernstein decomposition, it's natural to wonder whether an analogue of Theorem 3.15 holds and, if so, whether it's useful to understanding the finite presentedness of $\mathcal{H}(G(F), I^{(1)})$.
- (2) What specifically goes wrong with Breuil's proof if you replace \mathbb{Q}_p by another field F (either p -adic or equicharacteristic)?

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