The Scholze-Shin conjecture for unramified unitary groups

Part I: the trivial endoscopy case

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December 9, 2019

[1]This work is a part of the project "Kapibara" and was partially supported by the funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 802787)

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Introduction and notation

Introduction

This paper is the first part of a series of paper whose goal is to explore to what extent the results of [Sch13b] can be extended to groups other than $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$.

More explicitly, in [Sch13b] Scholze is able to to show that the local Langlands conjecture for $\mathrm{GL}_n(F)$, where F is a finite extension of $\mathbb{Q}_p$, can be characterized by explicitly constructed 'test functions'. Less cryptically, he shows that for every cutoff function $h \in C_c^\infty(\mathrm{GL}_n(\mathcal{O}_F), \mathbb{Q})$ and every element $\tau \in W_F$, there is an explicitly defined function $f_{\tau,h} \in \mathscr{H}(\mathrm{GL}_n(F))$ with the property that for any irreducible smooth representation π_p of $\mathrm{GL}_n(F)$ that

$$\mathrm{tr}(f_{\tau,h} \mid \pi_p) = \mathrm{tr}(h \mid \pi_p)\,\mathrm{tr}(\tau \mid \mathsf{LL}(\pi_p)), \tag{1}$$

where LL is the local Langlands correspondence for $\mathrm{GL}_n(F)$ as in [HT01]. Moreover, Scholze shows that (1) uniquely characterizes the correspondence LL.

The function $f_{\tau,h}$ was constructed by Scholze in the earlier work [Sch13a] and can be defined in terms of the cohomology of certain tubular neighborhoods inside of Rapoport-Zink spaces associated to $\mathrm{GL}_n(F)$. Note that, in particular, $f_{\tau,h}$ implicitly depends on the choice of a dominant cocharacter of $\mathrm{GL}_{n,F}$ which, in the above, is the cocharacter corresponding to the standard representation.

Scholze's function theoretic characterization of the local Langlands conjecture for $\mathrm{GL}_n(F)$ has many applications, examples of which we now list. Philosophically it suggests that the deep and somewhat abstract Langlands correspondence can be understood, in some sense, in terms of explicit functions which one might be able to algorithmically or combinatorially describe. A function theoretic characterization of the Langlands correspondence allows for a more concrete study of the endoscopic case of the Langlands functoriality principle, by studying the transfer of these characterizing functions between endoscopic groups. Finally, the function theoretic characterization of the local Langlands conjecture lends itself to be used to study the Langlands correspondence in more fluid situations (for example to study the local Langlands correspondence in families as in [JNS17]).

Given the above, especially in any attempt to study functoriality using these 'test functions', one desires to generalize this result of Scholze to an arbitrary reductive group G over $\mathbb{Q}_p$. In [SS13] Scholze and S.W. Shin study the cohomology groups $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ where Sh is the Shimura variety attached to certain compact unitary similitude groups $\mathbf{G}$ (those with no endoscopy as in §I.5). In particular, they describe the decomposition of the $\mathbf{G}(\mathbb{A}_f) \times W_{E_\mathfrak{p}}$ $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$, where E is the reflex field for Sh and $\mathfrak{p}$ is a prime of E lying over a *split place* p of $\mathbb{Q}$ (see loc. cit. for the definition of split, which is slightly less restrictive than the usual notion of split), in terms of the local

Langlands conjecture of $\mathbf{G}(\mathbb{Q}_p)$ which is (a product of terms of the form) $\mathrm{GL}_n(F)$.

They also formulate generalizations of the formula (1) to groups G over $\mathbb{Q}_p$ other than $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$. In particular, they state the following:

Conjecture 1 (Scholze-Shin)**.** *Let G be an unramified group over $\mathbb{Q}_p$ with $\mathbb{Z}_p$-model $\mathcal{G}$ and let μ be a dominant cocharacter of $G_{\overline{\mathbb{Q}_p}}$ with reflex field E. Let $\tau \in W_{\mathbb{Q}_p}$ and let $h \in C_c^\infty(\mathcal{G}(\mathbb{Z}_p), \mathbb{Q})$. Let (H, s, η) be an endoscopic group for G and let h^H be the transfer of h. Then, for every tempered L-parameter φ with associated semi-simple parameter λ we have*

$$S\Theta_\varphi(f_{\tau,h}^H) = \mathrm{tr}\left(s^{-1}\tau \mid (r_{-\mu} \circ \eta \circ \lambda\,|_{W_E}\, |\cdot|_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\varphi(h). \tag{2}$$

We refer the reader to [SS13, §7] for a detailed explanation of the notation but we note that $S\Theta_\varphi$ is the stable distribution of φ which associates to a function $f \in \mathscr{H}(H(\mathbb{Q}_p))$ the quantity

$$S\Theta_\varphi(f) := \sum_{\pi_p \in \Pi(\varphi)} r_\pi\, \mathrm{tr}(f \mid \pi_p), \tag{3}$$

where $\Pi(\pi_p)$ is the L-packet of φ and r_π is a natural number associated to π (see [SS13, §6]).

Remark. As remarked before, the function $f_{\tau,h}$ depends on the choice of μ, but we suppress this dependency throughout this article since it will always be clear from context.

Note that to make sense of Conjecture 1 one must have the analogue of the functions $f_{\tau,h}$ for G as well as the knowledge of the local Langlands conjecture for H. In this conjecture we are concerned with the case where $H = G$. In this case, the existence of the functions $f_{\tau,h}$ follows from the results of [You19] and the local Langlands conjecture for H follows from the results of [Mok15].

The desire for the presence of endoscopic groups in Conjecture 1 is related to the fact that to characterize the local Langlands conjecture for groups G different from $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$, for which non-trivial L-packets appear, one expects the need to relate any association with endoscopic transfer, which the necessitates a formula like Equation (2) for an arbitrary endoscopic group H.

The result of the methods in this paper is the following (stated as Theorem III.4.1) in the body of the paper):

Theorem 1. *The Scholze-Shin conjecture holds with the following assumptions:*

1. *$G = \mathsf{Res}_{F/\mathbb{Q}_p}U$ where U is an inner form of $U_{E/F}(n)^*$ and $E/\mathbb{Q}_p$ is unramified.*

2. *The parameter ψ is tempered.*

3. *The L-packet of ψ contains a square integrable representation.*

4. *(H, s, η) is the trivial endoscopic triple, and μ is miniscule*

Remark. We hope that the assumption that μ is miniscule is not crucial, and that it will be removed in a future draft of this article. The ability to remove this assumption, if possible, is due to the fact that the miniscule cocharacters of $\mathrm{GL}_n(\mathbb{C})$ generate the space of weights. Note that this is a special feature of $\mathrm{GL}_n(\mathbb{C})$ and, in particular, may be impossible to remove (with the techniques of this paper) for analogous results of Theorem 1 for other groups G.

Remark. In fact, we prove the above result for all local A-parameters ψ containing a representation π_p appearing as a local constituent of a representation π appearing in the cohomology of the unitary Shimura varieties we consider and such that π_∞ is discrete series.

We now describe the contents of our paper, pointing out interesting results which are incidental to the proof of Theorem 1.

Part I In Part I of the paper we explore the notion of *relevant endoscopy.* Informally speaking, the relevant endoscopy of a global group $\mathbf{G}$ is the set of endoscopic triples showing up in the stabilization of the trace formula for $\mathbf{G}$. More rigorously, we define an endosopic triple $(\mathbf{H}, s, \eta)$ to be *relevant* if it can be completed to an endoscopic quadruple $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}})$ (as in Definition I.2.4). We show that this notion of relevance is intimately related to an *a priori* different notion of relevance for (H, s, η) which means that it can be upgraded to a quadruple $(H, s, {}^L\eta, \psi^H)$ where ψ^H is an A-parameter for H and ${}^L\eta \circ \psi^H$ is relevant for G.

Remark. Here our notion of A-parameter is somewhat loose. In Part I we develop a method to analyze the above when the A-parameters of an algebraic group G over a local or global field F is taken to mean certain homomorphisms $\psi : \mathcal{L}_\psi \to {}^LG$ where $\mathcal{L}_\psi$ is some extension of W_F by a pro-reductive connected algebraic group. In particular, we shall apply this in the cases when F is local (in which case these are the usual notion of A-parameters) and when G is a global unitary group in which case they are the A-parameters in [KMSW14, §1.3.4].

This then allows one to get a good understanding of the explicit relationship between a unitary group $\mathbf{G}$ having no relevant endoscopy and certain global parameters ψ of $\mathbf{G}$ (as in [KMSW14]) having trivial reduced global centralizer group $\overline{S_\psi}$. Namely, we show the following (labeled as Proposition I.6.2 in the main body of the paper):

Theorem 2. *Let $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ be a global unitary group and let ψ be a relevant A-parameter of $\mathbf{G}$ such that ψ_∞ is elliptic for some infinite place ∞ of F. Then, if $\mathbf{G}$ has no relevant endoscopy then $\overline{S_\psi} = 1$.*

As a corollary of this, using the deep work of [KMSW14], we obtain, using the notation of Theorem 2, the following (labeled as Lemma I.6.3 in the main body of the paper):

Corollary 1. *Let π be an automorphic representation for $\mathbf{G}$ which is discrete at infinity. Then, if $\mathbf{G}$ has no relevant endoscopy the following equality holds*

$$L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\pi^p] = \bigoplus_{\pi'_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p),\omega_p)} \pi'_p \tag{4}$$

where ψ is the A-parameter associated to π.

For a precise description of notation see the discussion surrounding Lemma I.6.3. In words, this lemma says that under suitable conditions on $\mathbf{G}$ and π the away-from-p isotypic component of $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ associated to π consists of precisely representations with local p-component lying in the packet of ψ_p and, moreover, that these appear with multiplicity one.

Part II In Part II of this paper we show a decomposition of the cohomology of a compact Shimura variety with no endoscopy. More precisely, we have the the following (labeled as Theorem II.2 in the main body of the paper):

Theorem 3. *Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ which has no relevant endoscopy and for which $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Suppose that Sh is a Shimura variety associated $\mathbf{G}$ with reflex field $\mathbf{E}_{\boldsymbol{\mu}}$. Then, for any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of $\mathbf{G}$ and any prime $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ there is a decomposition of virtual $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f), \tag{5}$$

where π_f ranges over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ such that there exists an automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that;

1. *$\pi_f \cong (\pi)_f$ (using our identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)*
2. *$\pi_\infty \in \Pi_\infty(\xi)$.*

Moreover, for each π_f there exists a cofinite set $S(\pi_f) \subseteq S^{\mathrm{ur}}(\pi_f)$ of primes p such that for each prime $\mathfrak{p}$ over $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p and each $\tau \in W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ the following equality holds:

$$\mathrm{tr}(\tau \mid \sigma(\pi_f)) = a(\pi_f)\,\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ \varphi_{\pi_p}) p^{\frac{1}{2}v(\tau)[\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}:\mathbb{Q}_p]\dim \mathsf{Sh}}, \tag{6}$$

for some integer $a(\pi_f)$ (see Definition II.3.5).

Besides the singling out of the notion of relevance of endoscopy this theorem has minimal original content, essentially being a technical exercise in showing that the results of [Kot92a] are applicable to the general situation with the results of [KSZ] as a replacement for the results of [Kot92b]. We have included the work here mostly for the convenience of the reader, and to help fix ideas and notation that occur in Part III of the paper.

Part III In Part III we combine the results of the last two parts, together with the work of [Shi11] and [You19], to deduce Theorem 1.

To begin, we show that one can make explicit improvements to Theorem 3 in the case that $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ for a unitary group $\mathbf{U}$. Namely, we show the following (see the contents of §III.2):

Theorem 4. *Let $E/\mathbb{Q}$ be a CM field with F its totally real subfield. Let $\mathbf{U}$ be an inner form of $U_{E/F}(n)^*$ and set $\mathbf{G} := \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$. Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic and has no relevant endoscopy. Let Sh be a Shimura variety associated to $\mathbf{G}$. Then, for any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ and* any prime *$\mathfrak{p}$ of E there is a decomposition of virtual $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}]$-modules*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi)(\chi) = \bigoplus_{\pi_f} \pi_f \boxtimes a(\pi_f)\,(r_{-\boldsymbol{\mu}} \circ \mathsf{LL}(\pi_p))\,, \tag{7}$$

where π_f ranges over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ such that there exists an automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that;

1. *$\pi_f \cong (\pi)_f$ (using our identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)*
2. *$\pi_\infty \in \Pi_\infty(\xi)$.*

and χ is some global character and $a(\pi_f)$ is an integer (see Definition II.3.5).

We also obtain, using Theorem 4 and Corollary 1, the further refinement:

Corollary 2. *Let π be be an automorphic representation of $\mathbf{G}$ such that π_∞ is discrete series. Then, for any prime $\mathfrak{p}$ of E and any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ we have a decomposition of virtual $\mathbb{Q}_\ell[\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbf{E}_{\mu_\mathfrak{p}}}]$-modules*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi)[\pi_f^p] = \bigoplus_{\pi'_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \omega_p)} \pi'_p \boxtimes \sigma(\pi_f^p \otimes \pi'_p). \tag{8}$$

We then use the trace formula in [You19] together with Theorem 4 and Corollary 2 to deduce Theorem 1. To do this though, one must first lift local representations at p to global representations of some unitary group, and some care must be chosen in the conditions necssary to do this. We appeal to the results of [Shi12] which is where the square-integrability conditions enter into the equation.

Remark. The authors would like to point out that while much of the paper is written with the specific focus on unramified unitary groups, the rough strategy to prove the Scholze-Shin conjecture seems applicable to a much wider class of groups. The main impediments to generalizing is the lack of results like [KMSW14] and [Shi11] to apply to non-unitary groups.

Future directions

The authors intend to work on extending Theorem 1 to the case of an arbitrary endoscopic triple (H, s, η). While the authors are hopeful, this will be a serious undertaking. The main obstruction being the lack of a simple analogues of Theorem 3 and Corollary 1 in the situation of groups $\mathbf{G}$ which have non-trivial endoscopy.

Beyond that, the authors are interested in studying to what extent the Scholze-Shin conjecture characterizes the local Langlands correspondence and, in particular, in the situation of unramified unitary groups. The result for $\mathrm{GL}_n(F)$ in [Sch13b] uses results which don't obviously generalize to groups other than $\mathrm{GL}_n(F)$.

Acknowledgements

The authors began this work while graduate students at Berkeley under the supervision of Sug Woo Shin. They would like to thank him for his extensive guidance and support during the writing of this paper.

The authors would also like to thank Brian Conrad for helpful discussions.

Finally, a substantial portion of this paper was written at Robert and Stacey Youcis's home in Pennsylvania, USA. The authors heartily thank Mr. and Mrs. Youcis for their warm hospitality and for keeping a full pantry.

This work was partially supported by NSF RTG grant DMS-1646385.

Notations and conventions

General

- Unless stated otherwise p is a prime and ℓ is a prime different from p.
- We will (sometimes implicitly) fix an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \xrightarrow{\approx} \mathbb{C}$.
- Unless stated otherwise all fields are assumed of characteristic 0.
- For a number field F and a finite place v of F we shall denote by F_v the completion of F at v, $\mathcal{O}_v$ its integer ring, and k_v its residue field.

- For a number field F we denote by $\mathbb{A}_F$ the topological ring of F-adeles and by $\mathbb{A}_{F,f}$ the topological subring of finite F-adeles. We shall shorten $\mathbb{A}_{\mathbb{Q}}$ to $\mathbb{A}$ and $\mathbb{A}_{\mathbb{Q},f}$ to $\mathbb{A}_f$.

Galois theory

- For a field F and an algebraic extension F'/F we shall use $\mathrm{Gal}(F'/F)$ to denote the Galois group of F' over F. We shall shorten $\mathrm{Gal}(\overline{F}/F)$ to Γ_F.

- For a local or global field F we shall denote by W_F the Weil group of F (as in [Tat79, §1]) with its implicit continuous map with dense image $W_F \to \Gamma_F$. For every finite Galois extension F' of F we shall use this map to canonically, and implicitly, define an isomorphism $W_F/W_{F'} \cong \Gamma_F/\Gamma_{F'}$ and shall thus use $\mathrm{Gal}(F'/F)$ to denote the common group.

- For a non-archimedean local field F with residue field k we shall shall denote by $I_F \subseteq W_F \subseteq \Gamma_F$ the inertia subgroup of F.

- For a finite field F we shall denote by Frob_F, or just Frob if F is clear from context, the geometric Frobenius element in Γ_F.

- For a non-archimedean local field F with residue field k we shall denote by Frob_F a lift of Frob_k along the canonical surjection $W_F \to \Gamma_k$.

- For a local field F we shall denote by v_F, or just v when F is clear from context, the valuation map $v : W_F \to \mathbb{Z}$ where we have normalized so that $v(\mathrm{Frob}_F) = 1$.

Reductive groups

- All reductive groups are assumed connected.

- In contexts revolving arbitrary fields F we shall denote algebraic groups over F with non-boldfaced letters like G. In the context where F is a global field we will often denote a group over F in the boldface font (e.g. $\mathbf{G}$). For a place v of F we shall denote shorten $\mathbf{G}_{\mathbb{Q}_v}$ to $\mathbf{G}_v$. If there is some distinguished place v_0 of F of interest to us we shall often use the non-boldfaced notation G to denote $\mathbf{G}_{v_0}$.

- For an algebraic group G over a field F we denote by G° the connected component of G and by $\pi_0(G)$ the component group G/G°.

- For an algebraic group G over a field F we denote by $Z(G)$ the center of G and by $Z_G(\gamma)$ the centralizer of an element $\gamma \in G(F)$.

- For an algebraic group G over a field F and an element $\gamma \in G(F)$ we denote by I_γ the group $Z_G(\gamma)^\circ$.

- For an algebraic group G we denote $G/Z(G)$ by G^{ad} and the derived subgroup by G^{der}.

- For a reductive group G over a field F we denote by A_G the maximal F-split torus in $Z(G)$.

- For a reductive group G over a field F we shall denote by $X_*(G)$ the Γ_F-set of homomorphisms $\mathbb{G}_{m,\overline{F}} \to G_{\overline{F}}$ and by $X^*(G)$ the Γ_F-module of homomorphisms $G_{\overline{F}} \to \mathbb{G}_{m,\overline{F}}$. Note that if G is a torus then $X_*(G_{\overline{F}})$ is also a Γ_F-module. We denote by $X^*_F(G)$ the group of homomorphisms $\mathbb{G}_{m,F} \to G$ and identify it implicitly with the subgroup $X^*(G)^{\Gamma_F}$ of $X^*(G)$.

- For a reductive group G over a field F we denote by $\{G\}$ the set of conjugacy classes in $G(F)$, by $\{G\}_s$ the set of stable conjugacy classses in $G(F)$, and by $\{G\}^{\text{s.s.}}$ and $\{G\}_s^{\text{s.s.}}$ the analogues with $G(F)$ replaced by the set $G(F)^{\text{s.s.}}$ of semisimple elements of $G(F)$. For an element $\gamma \in G(F)$ we denote by $\{\gamma\}$ (resp. $\{\gamma\}_s$) its image in $\{G\}$ (resp. $\{G\}_s$).

- For a reductive group G over a field F and two elements γ and γ' in $G(F)$ we use the notation $\gamma \sim \gamma'$ to indicate that γ and γ' are conjugate, and the notation $\gamma \sim_{\text{st}} \gamma'$ to denote that γ and γ' are stably conjugate.

- For a reductive group G over a field F and a semi-simple element $\gamma \in G(F)$ we denote by $S(\gamma)$ the collection of conjugacy classes contained in the stable conjugacy class $\{\gamma\}_s$.

- For a reductive group G over a local field F and a semi-simple element $\gamma \in G(F)$ we denote by $a(\gamma)$ the cardinality of the kernel of the natural map
$$H^1(F, I_\gamma) \to H^1(F, Z_G(\gamma)) \tag{9}$$
which is finite by the assumption that F is local. Note that if G^{der} is simply connected then $a(\gamma) = 1$ and so this term will often times not factor in to our work (despite its presence in many references).

- For a reductive group G over a field F we denote by $G(F)^{\text{ell}}$ the set of elliptic elements of $G(F)$ (see §IV.1.1 for a discussion of ellipticity).

- If G is an algebraic group over a characteristic 0 local field we will topologize $G(F)$ in the standard way (e.g. as in [Con12b]). We shall then denote the connected component of $G(F)$ with this topology by $G(F)^0$.

- If F is a global field and $\mathbf{G}$ a reductive group over F we shall topologize $G(\mathbb{A}_F)$ and $\mathbf{G}(\mathbb{A}_{F,f})$ in the standard ways (again see [Con12b]) .

- For a number field F and a reductive group $\mathbf{G}$ over F we denote by $S(\mathbf{G})$ the set of finite places v of F for which $\mathbf{G}_v$ is unramified (i.e. which admits a reductive model over $\operatorname{Spec}(\mathcal{O}_v)$ in the sense of [Con14, Definition 3.1.1]).

- For a number field F and a reductive group $\mathbf{G}$ over F we will often implicitly choose a reductive model $\mathcal{G}_v$ of $\mathbf{G}_v$ over $\operatorname{Spec}(\mathcal{O}_v)$ for all $v \in S(\mathbf{G})$.

- We shall denote by $K_{0,v}$ the hyperspecial subgroup $\mathcal{G}_v(\mathcal{O}_v) \subseteq \mathbf{G}(F_v)$ for all $v \in S(\mathbf{G})$. For finite $v \notin S(\mathbf{G})$ or infinite v we shall define $K_{0,v}$ to be $\mathbf{G}(F_v)$.

- We will implicitly make the identification of topological groups

$$G(\mathbb{A}_F) \cong \prod_v{}' (\mathbf{G}(F_v), K_{0,v}) \tag{10}$$

and the identification

$$G(\mathbb{A}_{F,f}) \cong \prod_{v \text{ finite}}{}' (\mathbf{G}(F_v), K_{0,v}) \tag{11}$$

obtained by (passing to the colimit) in [Con12b, Theorem 3.6].

- For a reductive group over a number field F we denote by $\mathbf{G}(\mathbb{A}_F)^1$ the subgroup of $\mathbf{G}(\mathbb{A}_F)$ defined as follows

$$\mathbf{G}(\mathbb{A})^1 := \{g \in \mathbf{G}(\mathbb{A}) : |\nu(g)| = 1 \text{ for all } \nu \in X^*(\mathbf{G})^{\Gamma_F}\} \tag{12}$$

where $\mathbb{A}_F^\times$ is given the usual norm.

- For a reductive group $\mathbf{G}$ over the number field F we note that evidently (by the product rule) that $\mathbf{G}(F) \subseteq \mathbf{G}(\mathbb{A}_F)^1$ we define the *adelic quotient* of $\mathbf{G}$, denoted $[\mathbf{G}]$, to be the topological space $\mathbf{G}(\mathbb{A})^1/\mathbf{G}(\mathbb{Q})$ which is a measure space whenever $G(\mathbb{A})$ is given a measure.

- For F a global field and $\mathbf{G}$ a reductive group over F we denote by $\tau(\mathbf{G})$ the *Tamagawa number* of $\mathbf{G}$ defined to be $\operatorname{vol}([G])$ when $\mathbf{G}(\mathbb{A})$ is endowed with the Tamagawa measure (as in [Wei12, Chapter II]). See [PS92, Theorem 5.6] for a proof that such a volume is finite.

- For $\mathbf{G}$ a reductive group over $\mathbb{Q}$ and K a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ we denote by $Z(\mathbb{Q})_K$ the group $Z(\mathbf{G})(\mathbb{Q}) \cap K$ and by Z_K the group $Z(\mathbf{G})(\mathbb{A}_f) \cap K$.

- Let F be a local field and G a reductive group over F. We denote by $e(G)$ the *Kottwitz sign* as in [Kot83].

Harmonic analysis

- Let F be a number field and $\mathbf{G}$ a reductive group over F. Let C be an algebraically closed field and let π_f be an irreducible admissible C-representation of $\mathbf{G}(A_{F,f})$. Then, we shall denote by

$$\pi_f = \bigotimes_v{}' \pi_{f,v} \tag{13}$$

the Flath decomposition with respect to the set $\{K_{0,v}\}$ as in [Fla79]. We then denote by $S^{\mathrm{ur}}(\pi_f)$ the set of $v \in S(\mathbf{G})$ such that $\pi_{f,v}$ is $K_{0,v}$ unramified (i.e. for which $\pi_{f,v}^{K_{0,v}} \neq 0$) and call a place v in $S^{\mathrm{ur}}(\pi_f)$ *unramified*. Again, we will make it clear when things fundamentally change with different choices of $K_{0,v}$.

- If $v \in S^{\mathrm{ur}}(\pi_f)$ let us denote by $\varphi_{\pi_{f,v}}$ the associated unramified local Langlands parameter $W_{F_v} \to {}^L\mathbf{G}_v$ as in [Bor79, Chapter II].

- Let F be a non-archimedean local field and let G be a reductive group over F. For a characteristic 0 field C We denote by $\mathscr{H}_C(G(F))$, or just $\mathscr{H}(G(F))$ when C is clear the *Hecke algebra* as in [C$^+$79, §1.3] where we have implicitly (often times clear from context) fixed a $\mathbb{Q}$-valued Haar measure dg on $G(F)$. For a compact open subgroup K of $G(F)$ we shall denote by $\mathscr{H}_C(G(F), K)$, or just $\mathscr{H}(G(F), K)$ when C is clear from context, as in loc. cit.

- Let F be a local field and G a reductive group over F. Let us suppose that $\phi \in \mathscr{H}_C(G(F))$ and that $\gamma \in G(F)$ is semi-simple. Then, we define the *orbital integral* of ϕ, denoted $O_\gamma(\phi)$, to be the quantity

$$O_\gamma(\phi) := \int_{I_\gamma(F)\backslash G(F)} \phi(g\gamma g^{-1})\, dg \tag{14}$$

We define the *stable orbital integral* of ϕ, denoted $SO_\gamma(\phi)$, to be the quantity

$$SO_\gamma(\phi) = \sum_{\gamma' \sim_{\mathrm{st}} \gamma} e(I_{\gamma'}) a(\gamma') O_\gamma(\phi) \tag{15}$$

- Let F be a global field and let $\mathbf{G}$ be a reductive group over F. Let ϕ be an element of $\mathscr{H}_C(\mathbf{G}(\mathbb{A}_F))$ and $\gamma \in \mathbf{G}(\mathbb{A}_F)$ semi-simple (i.e. that each of its local factors is semi-simple). We then define the *orbital integral* of ϕ, denoted $O_\gamma(\phi)$, to be the quantity

$$O_\gamma(\phi) = \int_{\mathbf{I}_\gamma(\mathbb{A}_F)\backslash \mathbf{G}(\mathbb{A}_F)} \phi(g\gamma g^{-1})\, dg \tag{16}$$

Assume now that $\gamma \in \mathbf{G}(F)$. We define the *stable orbital integral* of ϕ, denoted $SO_\gamma(\phi)$, to be the quantity

$$\sum_i e(\mathbf{I}_{\gamma_i})O_{\gamma_i}(\phi) \tag{17}$$

Here i ranges over the set

$$\ker(F, \mathbf{I}(\overline{\mathbb{A}_F})) \to H^1(F, \mathbf{G}(\overline{\mathbb{A}_F})) \tag{18}$$

The element $\gamma_i \in \mathbf{G}(\mathbb{A}_F)$ is the one associated to i by applying [Kot86b, §4.1] place by place. Note, in particular, that for all places v of F the v^{th}-component of γ_i is stably conjugate to γ.

- Suppose that $\mathbf{G}$ is a reductive group over $\mathbb{Q}$ and $\xi_\mathbb{C}$ is an algebraic representation of $\mathbf{G}_\mathbb{C}$. Let $\Pi_\infty(\xi_\mathbb{C})$ be the set of isomorphism classes of all irreducible $\mathbf{G}(\mathbb{R})$-representations having the same central and infinitesimal character as the $\overline{\text{contragredient}}$ representation and let $\Pi^0_\infty(\xi_\mathbb{C})$ be the subset of discrete series representations in $\Pi_\infty(\xi_\mathbb{C})$. If ξ is an algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbf{G}$ we use our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$ to obtain a corresponding $\mathbb{C}$-representation $\xi_\mathbb{C}$ and we set $\Pi_\infty(\xi) := \Pi_\infty(\xi_\mathbb{C})$ and $\Pi^0_\infty(\xi) := \Pi^0_\infty(\xi_\mathbb{C})$

- Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$. Let π be a $\mathbb{C}$-representation (or $\overline{\mathbb{Q}_\ell}$-representation using our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$). We set $m(\pi)$ to be the multiplicity of π in $L^2_{\text{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$.

Algebraic geometry

- For a variety X over a field k and a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X with $\text{char}(k) \neq \ell$ we then denote by $H^*(X, \mathcal{F})$ the virtual $\overline{\mathbb{Q}_\ell}$-space

$$\sum_{i=0}^{2\dim(X)} (-1)^i H^i(X_{\overline{k}}, \mathcal{F}_{\overline{k}}). \tag{19}$$

Shimura varieties

- We shall denote Shimura data as $(\mathbf{G}, X)$ as in [Mil04, Definition 5.5].
- We shall assume that all of our Shimura data are of abelian type.
- We shall assume only that our Shimura data satisfy axioms **SV**1, **SV**2, and **SV**3 as in [Mil04], but will often assume that our Shimura data also satisfies axiom of **SV**5.
- If $(\mathbf{G}, X)$ is a Shimura datum, we shall denote its associated reflex field (as in [Mil04, Definition 12.2]) by $E(\mathbf{G}, X)$ or, when $(\mathbf{G}, X)$ is clear from context, just E.

- For every neat (as on [Mil04, Page 34]) compact open subgroup K of $\mathbf{G}(\mathbb{A}_f)$ we denote by $\mathsf{Sh}_K(\mathbf{G}, X)$, or Sh_K when $(\mathbf{G}, X)$ is clear from context, the canonical model (in the sense of [Mil04, Definition 12.8]) of the complex variety $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ (as in [Mil04, Definition 5.14]) over its reflex field E.

- We denote by Sh the E-scheme $\varprojlim_K \mathsf{Sh}_K$ as K runs over the neat compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$. Note that this exists by [Sta18, Tag 01YX] since the transition maps for the system $\{\mathsf{Sh}_K\}$ have finite (and thus affine) transition maps.

- Let ℓ be a prime and let ξ be an algebraic $\overline{\mathbb{Q}_\ell}$-representation of G (i.e. an algebraic representation $\xi : G_{\overline{\mathbb{Q}_\ell}} \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ for some $\overline{\mathbb{Q}_\ell}$-space V) such that for the induced map

$$G(\mathbb{A}_f) \xrightarrow{\text{proj.}} G(\mathbb{Q}_\ell) \hookrightarrow G(\overline{\mathbb{Q}_\ell}) \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$$

has the property that $Z(\mathbb{Q})_K \subseteq \ker \xi$ for all sufficiently small compact open subgroups $K \subseteq G(\mathbb{A}_f)$.

Part I

Relevant global endoscopy

I.1 Introduction

In this part, we discuss the notion of relevant global endoscopy. Loosely, for a group $\mathbf{G}$ defined over a number field F, we say that an elliptic endoscopic datum $(\mathbf{H}, s, \eta)$ is *relevant* if it appears in the stable trace formula for the group $\mathbf{G}$. We then prove some applications of our discussion which will be necessary for our main results.

I.2 Definitions and statements

We assume for convenience in this entire part that G^{der} is simply connected. We begin by recalling the definition of endoscopic datum as in [Shi10, §2.1].

Definition I.2.1. *An endoscopic datum for a reductive group G over a field F consists of a triple (H, s, η) where H is a quasisplit reductive group, $\eta : \widehat{H} \to \widehat{G}$ is an embedding and $s \in \widehat{H}$ such that*

- *We have an equality $\eta(\widehat{H}) = Z_{\widehat{G}}(s)^0$,*
- *The $\widehat{G}$-conjugacy class of η is fixed by Γ_F,*
- *The image of s in $Z(\widehat{H})/Z(\widehat{G})$ lies in $(Z(\widehat{H})/Z(\widehat{G}))^{\Gamma_F}$,*
- *The image of $s \in H^1(F, Z(\widehat{G}))$ is trivial if F is local and locally trivial if F is global.*

An endoscopic datum is defined to be elliptic *if $(Z(\widehat{H})^{\Gamma})^{\circ} \subset Z(\widehat{G})$.*

We record now our definition of isomorphism between endoscopic data:

Definition I.2.2. *An isomorphism between endoscopic data (H_1, s_1, η_1) and (H_2, s_2, η_2) is an isomorphism $\alpha : H_2 \to H_1$ such that there exists $g \in \widehat{G}$ such that $\widehat{\alpha}(s_1) = s_2 \mod Z(\widehat{G})$ and the following diagram commutes:*

$$\begin{tikzcd} \widehat{H_1} \arrow[r, "\eta_1"] \arrow[d, "\widehat{\alpha}"'] & \widehat{G} \arrow[d, "\mathrm{Int}(g)"] \\ \widehat{H_2} \arrow[r, "\eta_2"] & \widehat{G}. \end{tikzcd} \tag{20}$$

We denote the set of isomorphism classes of endoscopic data for G by $\mathcal{E}(G)$ and we denote the set of isomorphism classes of elliptic endoscopic data by $\mathcal{E}^{\mathrm{ell}}(G)$.

Note that the map $\widehat{\alpha}$ is Γ_F-invariant and only well-defined up to a choice of splittings (see [Kot84b, §1.8]) and hence up to $\widehat{H_1}^{\Gamma_F}$-conjugacy but that the above diagram makes sense for any choice of $\widehat{\alpha}$ in this class. Note also that we will often confuse $\widehat{H}$ for $\eta(\widehat{H})$ and so, in particular, will often confuse s and $\eta(s)$.

Now, since we assume G^{der} is simply connected, for each endoscopic datum (H, s, η), there exists a lift of η to an L-map ${}^L\eta : {}^LH \to {}^LG$ (see [Lan79, Prop 1]). The following lemma will be useful to us.

Lemma I.2.3. *Suppose that (H_1, s_1, η_1) and (H_2, s_2, η_2) are endoscopic data and fix lifts ${}^L\eta_1$ and ${}^L\eta_2$ of η_1 and η_2 respectively. Suppose further that $\alpha : H_2 \to H_1$ gives an isomorphism of endoscopic data and $g \in \widehat{G}$ is as in I.2.2. Then for each choice of $\widehat{\alpha}$, there exists a lift ${}^L\alpha$ of α such that the following diagram commutes:*

$$\begin{tikzcd}
{}^LH_1 \arrow[r, "{}^L\eta_1"] \arrow[d, "{}^L\alpha"'] & {}^LG \arrow[d, "\mathrm{Int}(g)"] \\
{}^LH_2 \arrow[r, "{}^L\eta_2"] & {}^LG.
\end{tikzcd} \tag{21}$$

Moreover, the $\widehat{H_1}$-conjugacy class of ${}^L\alpha$ does not depend on the choice of $\widehat{\alpha}$ or g.

Proof. We want to define ${}^L\alpha$ to equal ${}^L\eta_2^{-1} \circ \mathrm{Int}(g) \circ {}^L\eta_1$. For this to make sense, we need to show that the image of $\mathrm{Int}(g) \circ {}^L\eta_1$ is contained in the image of ${}^L\eta_2$.

Now there exists for each $w \in W_F$ and $i \in \{1, 2\}$, elements $g(w)_i \in \widehat{G}$ so that ${}^L\eta_i(1, w) = (g(w)_i, w)$. We observe that for any $h_i \in \widehat{H_i}$, we have

$$\begin{aligned}
(g(w)_i(w \cdot \eta_i)(h_i), w) &= {}^L\eta_i(1, w)\, {}^L\eta_i(w^{-1}(h_i), 1) && (22)\\
&= {}^L\eta_i(h_i, w) && (23)\\
&= {}^L\eta_i(h_i, 1)\, {}^L\eta_i(1, w) && (24)\\
&= (\eta_i(h_i)g(w)_i, w), && (25)
\end{aligned}$$

so that

$$\mathrm{Int}(g(w)_i^{-1})(\eta_i(h_i)) = (w \cdot \eta_i)(h_i). \tag{26}$$

Now, it suffices to check that for each $(1, w) \in {}^LH_1$ there exists an $(h_2, w) \in {}^LH_2$ such that

$$(gg(w)_1 w(g^{-1}), w) = (\eta_2(h_2)g(w)_2, w). \tag{27}$$

Hence we need to check that $gg(w)_1w(g^{-1})g(w)_2^{-1} \in \eta_2(\widehat{H_2})$. It suffices to show that this element lies in $Z_{\widehat{G}}(\eta_2(\widehat{H_2}))$ since for any maximal torus T of $\widehat{H_2}$, we have $\eta_2(T)$ is a maximal torus of $\widehat{G}$ and so

$$Z_{\widehat{G}}(\eta_2(\widehat{H_2})) \subset Z_{\widehat{G}}(\eta_2(T)) = \eta_2(T) \subset \eta_2(\widehat{H_2}). \tag{28}$$

Now pick $h_2 \in \widehat{H_2}$. We observe that using equation (26), we have

$$\begin{aligned}
\mathrm{Int}(gg(w)_1 w(g^{-1})g(w)_2^{-1})(\eta_2(h_2)) &= \mathrm{Int}(gg(w)_1 w(g^{-1}))((w\cdot\eta_2)(h_2)) && (29)\\
&= \mathrm{Int}(gg(w)_1)(w(g^{-1}\eta_2(w^{-1}(h_2))g)) && (30)\\
&= \mathrm{Int}(gg(w)_1)(w(\eta_1(\widehat{\alpha}^{-1}(w^{-1}(h_2))))) && (31)\\
&= \mathrm{Int}(gg(w)_1)((w\cdot\eta_1)(\widehat{\alpha}^{-1}(h_2))) && (32)\\
&= \mathrm{Int}(g)(\eta_1(\widehat{\alpha}^{-1}(h_2))) && (33)\\
&= \eta_2(h_2), && (34)
\end{aligned}$$

as desired.

Now we show the second statement of the lemma. As above, we have that the map $\widehat{\alpha}$ is unique up to $\widehat{H_1}^{\Gamma_F}$-conjugacy. For a fixed choice of $\widehat{\alpha}$ if we have pick two different $g, g' \in \widehat{G}$ such that the requisite diagram commutes, then $\mathrm{Int}(g^{-1}g')$ fixes $\eta_1(\widehat{H_1})$ pointwise and so $g^{-1}g' \in \eta_1(Z(\widehat{H_1}))$. Hence any two ${}^L\alpha$ will differ at most up to conjugacy by an element of $\widehat{H_1}$. $\square$

We are now ready to define the notion of relevant endoscopy. We begin with some definitions following [Shi10, §2.3].

The first definition is that of the set of so-called *endoscopic quadruples* for the group G:

Definition I.2.4. *For F a local or global field define $\mathcal{EQ}_F(G)$ to be the set of equivalence classes of tuples (H, s, η, γ_H) such that (H, s, η) is an endoscopic triple and $\gamma_H \in H(F)$ transfers to $G(F)$ and is (G,H)-regular and semisimple. The tuples (H, s, η, γ_H) and $(H', s', \eta', \gamma'_H)$ are equivalent if there exists an isomorphism $\alpha : H' \to H$ inducing an isomorphism of endoscopic data and such that $\alpha(\gamma'_H)$ is stably conjugate to γ_H. We define the subset $\mathcal{EQ}_F^{\mathrm{ell}}(G) \subset \mathcal{EQ}_F(G)$ to consist of those equivalence classes such that (H, s, η) is elliptic.*

We now define a set of pairs associated to G consisting, essentially, of a semi-simple element γ of $G(F)$ and an element of its Kottwitz group $\mathfrak{K}(I_\gamma/F)$ (see IV.1.5 for a recollection of the Kottwitz group). More precisely:

Definition I.2.5. *For F a local or global field define $\mathcal{SS}_F(G)$ to be the set of equivalence classes of pairs (γ, κ) such that $\gamma \in G(F)$ is semisimple and $\kappa \in \mathfrak{K}(I_\gamma/F)$. Two pairs (γ, κ) and (γ', κ') are equivalent if γ and γ' are stably conjugate in G and κ and κ' are equal under the canonical isomorphism $\mathfrak{K}(I_\gamma/F) \cong \mathfrak{K}(I_{\gamma'}/F)$. We define the subset $\mathcal{SS}_F^{\mathrm{ell}}(G) \subset \mathcal{SS}_F(G)$ to be the equivalence classes of pairs where γ is elliptic.*

Now we have the following key bijection due to Kottwitz:

Proposition I.2.6. *The natural map*

$$\mathcal{EQ}_F(\mathbf{G}) \to \mathcal{SS}_F(\mathbf{G}), \tag{35}$$

given by

$$(H, s, \eta, \gamma_H) \mapsto (\gamma, \eta(s)), \tag{36}$$

(where γ is some transfer of γ_H to $\mathbf{G}(F)$) is well-defined and a bijection. Moreover this map restricts to give a bijection

$$\mathcal{EQ}_F^{\text{ell}}(\mathbf{G}) \to \mathcal{SS}_F^{\text{ell}}(\mathbf{G}). \tag{37}$$

Proof. See [Shi10, Lemma 2.8] as well as [Kot86b, Lemma 9.7]. □

We are now ready to define the notion of relevant endoscopy.

Definition I.2.7. *Let F be a number field and $\mathbf{G}$ a reductive group over F. We have a natural projection map*

$$\mathcal{EQ}_F(\mathbf{G}) \to \mathcal{E}(\mathbf{G}). \tag{38}$$

which restricts to a map

$$\mathcal{EQ}_F^{\text{ell}}(\mathbf{G}) \to \mathcal{E}^{\text{ell}}(\mathbf{G}). \tag{39}$$

We define the subsets $\mathcal{RE}(\mathbf{G}) \subset \mathcal{E}(\mathbf{G})$ and $\mathcal{RE}^{\text{ell}}(\mathbf{G}) \subset \mathcal{E}^{\text{ell}}(\mathbf{G})$ to be the images of the first and second maps respectively. We say that the set $\mathcal{RE}(\mathbf{G})$ is the set of relevant global endoscopy *of $\mathbf{G}$ and that $\mathcal{RE}^{\text{ell}}(\mathbf{G})$ is the set of* relevant elliptic global endoscopy.

We now state the representation-theoretic analogue of I.2.6, part of a general web of analogies between representation theory and conjugacy classes. Such constructions appear for instance in works of Kottwitz (see the proof of [Kot84b, Prop 11.3.2]) and Shelstad ([She83, §4.2]). We choose to provide the details in this work.

For the remainder of this section, let us fix F to be a local or global field and G a reductive group over F.

We shall use the notion of A-parameters which we now recall. To do this we will be using the notion of the Langlands group $\mathcal{L}_F$ as in the introduction of [Art02]. When F is a local field such a group is $W_F \times \mathrm{SL}_2(\mathbb{C})$ but when F is a number field the existence of such a Langlands group (for which we use Langlands original pro-algebraic formalism) is conjectural. We shall then only use its basic properties assumed for such a group as in loc. cit.

We shall denote by K the kernel of the projection map $\mathcal{L}_F \to W_F$ which is a connected pro-algebraic group over $\mathbb{C}$ (which we often tacitly identify with its $\mathbb{C}$-points).

We begin with the definition of an L-parameter since this will make the definition of an A-parameter easier to parse:

Definition I.2.8. *Let $\mathcal{L}_F$ be the Langlands group. Then, an L-*parameter *for G is a continuous map $\phi : \mathcal{L}_F \to {}^LG$ such that the following conditions hold:*

1. *The restriction of the map $\phi_{|K}$ has image in $\widehat{G} \subseteq {}^LG$ and is algebraic as a map $K \to \widehat{G}$.*

2. *The diagram*
$$\begin{tikzcd} \mathcal{L}_F \arrow[r, "\phi"] \arrow[dr] & {}^LG \arrow[d] \\ & W_F \end{tikzcd} \tag{40}$$
is commutative.

3. *For all $w \in \mathcal{L}_F$ the element $\phi(w) \in {}^LG$ is semisimple or, in other words, that under any representation ${}^LG \to \mathrm{GL}_n(\mathbb{C})$ (in the sense of [Bor79, §2.6]) the image of $\phi(w)$ is semi-simple.*

Two L parameters ϕ_1 and ϕ_2 for G are said to be equivalent *if there exists $g \in \widehat{G}$ such that*
$$w \mapsto g^{-1}\phi_2(w)g\phi_1(w)^{-1} \tag{41}$$
is a (locally) trivial 1 cocycle of $\mathcal{L}_F$ taking values in $Z(\widehat{G})$.

In the case that F is local, we say that the L-parameter ϕ is relevant *if whenever $\phi(\mathcal{L}_F) \subset \mathcal{P}$ for $\mathcal{P}$ a parabolic subgroup of ${}^LG$ (in the sense of [Bor79, §3]), then $\mathcal{P}$ is conjugate in ${}^LG$ to ${}^LP$ for some parabolic subgroup $P \subseteq G$. In the case that F is global, we say that ϕ is relevant if for each place v of F, we have $\phi_v := \psi|_{\mathcal{L}_{F_v}}$ is relevant.*

We then move on to the slight variant of L-parameters known as A-parameters:

Definition I.2.9. *Let $\mathcal{L}_F$ be the Langlands group. Then, an A-*parameter *for G is a continuous map $\psi : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ such that the following conditions hold:*

1. *The restriction $\psi_{|\mathcal{L}_F}$ is an L-parameter.*

2. *The restriction $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ takes image in $\widehat{G}$ and the resulting map of complex Lie groups is holomorphic.*

3. *The diagram*
$$\begin{tikzcd} \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\psi"] \arrow[dr] & {}^LG \arrow[d] \\ & W_F \end{tikzcd} \tag{42}$$
is commutative.

4. *The image of $\psi(\mathcal{L}_F)$ in ${}^LG$ is bounded (i.e. relatively compact).*

Two A parameters ψ_1 and ψ_2 for G are said to be equivalent *if there exists $g \in \widehat{G}$ such that*

$$w \mapsto g^{-1}\psi_2(w)g\psi_1(w)^{-1} \tag{43}$$

is a (locally) trivial 1 cocycle of $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ taking values in $Z(\widehat{G})$.

In the case that F is local, we say that the A-parameter ψ is relevant *if whenever $\psi(\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})) \subset \mathcal{P}$ for $\mathcal{P} \subset {}^LG$ a parabolic subgroup, then $\mathcal{P}$ is conjugate in ${}^LG$ to ${}^LP$ for some parabolic subgroup $P \subseteq G$. In the case that F is global, we say that ψ is relevant if for each place v of F, we have $\psi_v := \psi|_{\mathcal{L}_{F_v} \times \mathrm{SL}_2(\mathbb{C})}$ is relevant.*

We also need the notion of when, for (H, s, η) an endoscopic triple for G, two A-parameters ψ_1^H and ψ_2^H of H are $Z(\widehat{G})$-equivalent. This definition is as follows:

Definition I.2.10. *Let (H, s, η) and endoscopic group of G. Then, two A-parameters ψ_1^H and ψ_2^H of H are said to be* $Z(\widehat{G})$-equivalent *if there exists an element $h \in \widehat{H}$ such that the map*

$$w \mapsto h^{-1}\psi_2^H(w)h\psi_1^H(w)^{-1}, \tag{44}$$

is a (locally) trivial 1-cocycle of $\mathcal{L}_F \times SL_2(\mathbb{C})$ valued in $Z(\widehat{G})$.

We need the following definitions as in [Kot84b, §10].

Definition I.2.11. *Let G be a reductive group over F and let ψ be an A parameter for G. Then we define C_ψ to be the set of $g \in \widehat{G}$ such that g commutes with the image of ψ. We also define S_ψ as the set of $g \in \widehat{G}$ such that*

$$w \mapsto g^{-1}\psi(w)g\psi(w)^{-1}, \tag{45}$$

is a (locally) trivial 1-cocycle of $\mathcal{L}_F \times SL_2(\mathbb{C})$ valued in $Z(\widehat{G})$. Note that evidently $Z(\widehat{G}) \subseteq S_\psi$ and we define $\overline{S_\psi}$ to be $S_\psi/Z(\widehat{G})$.

We define an A-parameter ψ to be *elliptic* if ψ factors through no proper Levi subgroup of ${}^LG$ and we have the following lemma of Kottwitz

Lemma I.2.12. *The following are equivalent.*

1. *The parameter ψ is elliptic,*
2. *$C_\psi^\circ \subset Z(\widehat{G})$,*
3. *$S_\psi^\circ \subset Z(\widehat{G})$.*

Proof. See [Kot84b, Lemma 10.3.1]. □

We now move towards stating our desired bijection. We begin first by defining the set on one side of the bijection. Roughly, this consists of A-parameters for endoscopic groups for G. More precisely:

Definition I.2.13. *Define the set $\mathcal{EP}_F(G)$ to be equivalences classes of quadruples $(H, s, {}^L\eta, \psi^H)$ where ${}^L\eta : {}^LH \to {}^LG$ is an L-map, $(H, s, {}^L\eta|_{\widehat{H}})$ is an endoscopic datum, and ψ^H is an A-parameter of H such that ${}^L\eta \circ \psi^H$ is relevant.*

Two quadruples $(H_1, s_1, {}^L\eta_1, \psi_1^H)$ and $(H_2, s_2, {}^L\eta_2, \psi_2^H)$ are equivalent if there is an isomorphism $\alpha : H_2 \to H_1$ of endoscopic data such that ${}^L\alpha \circ \psi_1^H$ is $Z(\widehat{G})$-equivalent to ψ_2^H. By I.2.3, note that the choice of ${}^L\alpha$ is unique up to $\widehat{H_1}$-conjugacy and that the notion of $Z(\widehat{G})$ equivalence does not depend on this choice.

We define $\mathcal{EP}_F^{\mathrm{ell}}(G) \subset \mathcal{EP}_F(G)$ to be the subset consisting of those tuples such that (H, s, η) is an elliptic endoscopic datum and ${}^L\eta \circ \psi^H$ is elliptic.

We then have the following definition of the other set in our desired bijection:

Definition I.2.14. *Define the set $\mathcal{SP}_F(G)$ of equivalence classes of pairs $(\psi, \overline{s})$ such that ψ is a relevant Arthur parameter of G and $\overline{s} \in \overline{S_\psi}$. Two pairs $(\psi_1, \overline{s}_1)$ and $(\psi_2, \overline{s}_2)$ are equivalent if ψ_1 and ψ_2 are equivalent by some $g \in \widehat{G}$ such that $\mathrm{Int}(g)(\overline{s}_1)$ and $\overline{s}_2$ are conjugate in $\overline{S_{\psi_2}}$.*

We define $\mathcal{SP}_F^{\mathrm{ell}}(G) \subset \mathcal{SP}_F(G)$ to consist of those pairs such that ψ is elliptic.

We can now finally state our desired bijection:

Proposition I.2.15. *The map*

$$[H, s, {}^L\eta, \psi^H] \mapsto [{}^L\eta \circ \psi^H, \overline{\eta(s)}] \tag{46}$$

gives a well-defined bijection $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$. Moreover, this map restricts to a bijection

$$\mathcal{EP}_F^{\mathrm{ell}}(G) \to \mathcal{SP}_F^{\mathrm{ell}}(G). \tag{47}$$

We now consider the case where F is a global field and $\mathbf{G}$ is a reductive group over F. We have another construction analogous to that of $\mathcal{RE}(\mathbf{G})$ and $\mathcal{RE}^{\mathrm{ell}}(\mathbf{G})$. Namely we define $\mathcal{REP}(\mathbf{G})$ to be the image of the projection

$$\mathcal{EP}_F(\mathbf{G}) \to \mathcal{E}(\mathbf{G}), \tag{48}$$

and $\mathcal{REP}^{\mathrm{ell}}(\mathbf{G})$ to be the image of the projection

$$\mathcal{EP}_F^{\mathrm{ell}}(\mathbf{G}) \to \mathcal{E}(\mathbf{G}). \tag{49}$$

This suggests the following

Question I.2.16. Is it true that

$$\mathcal{REP}(\mathbf{G}) = \mathcal{RE}(\mathbf{G}), \tag{50}$$

and

$$\mathcal{REP}^{\text{ell}}(\mathbf{G}) = \mathcal{RE}^{\text{ell}}(\mathbf{G})? \tag{51}$$

An important remark to make is that the previous discussion as well as the statement of I.2.15 for global F are contingent on the definition of the global Langlands group $\mathcal{L}_F$. In fact, our proof of I.2.15 uses this group in a somewhat nontrivial way, as we need to use ψ to construct a Galois action on $\widehat{H}$. We instead we prove the following result, which can be seen as evidence of the conjectured inclusion $\mathcal{REP}^{\text{ell}}(\mathbf{G}) \subset \mathcal{RE}^{\text{ell}}(\mathbf{G})$. This result carries no hidden conjectures on the Langlands correspondence. In particular, we will use it in the proof of our main result on the Scholze-Shin conjecture.

Theorem I.2.17. *Suppose that F is a totally real number field. Suppose that we have a triple $(\mathbf{H}, s, {}^L\eta)$ such that $(\mathbf{H}, s, \eta)$ is an endoscopic group for $\mathbf{G}$ and ${}^L\eta$ is an extension of η to ${}^LH$. In particular, for each place v of F we get an endoscopic datum $(\mathbf{H}_v, s, {}^L\eta_v)$ of $\mathbf{G}_v$. Suppose further that for each place v, we have an A-parameter $\psi_v^{\mathbf{H}}$ of $\mathbf{H}_v$ such that ${}^L\eta_v \circ \psi_v^{\mathbf{H}}$ is relevant. We assume further that at each real place v_∞, $(\mathbf{H}_{v_\infty}, s, \eta)$ is elliptic and that $\mathbf{H}_{v_\infty}$ has an elliptic maximal torus. Then in fact $(\mathbf{H}, s, \eta) \in \mathcal{RE}(\mathbf{G})$.*

Remark I.2.18. The restriction that F is totally real is not really a strong condition since it is almost implied by the later assumptions. In particular, to have that $\mathbf{H}_{V_\infty}$ has an elliptic maximal torus for all infinite places v_∞ implies, unless $\mathbf{H}$ is itself a torus, that F is totally real.

I.3 Proof of I.2.15

We now give the proof of the key bijection I.2.15. Before we begin the proof in earnest, it will be helpful to establish two useful general lemmata.

The first is the following:

Lemma I.3.1. *Let X be a complex reductive group. Let $s \in X(\mathbb{C})$ be semisimple and set $Y := Z_X(s)^\circ$. Then, the map $N_X(Y) \to \mathrm{Out}(Y)$ given on $\mathbb{C}$-points by sending $x \in N_X(Y)(\mathbb{C})$ to $\mathrm{Int}(x)_{|Y}$ has finite image.*

Proof. Let us note that $Z_X(Z(Y))^\circ$ is contained in the kernel of the map $N_X(Y) \to \mathrm{Out}(Y)$. Indeed, it suffices to show that $Z_X(Z(Y))^\circ \subseteq Y$. We first observe that $s \in Z(Y)$. Evidently $s \in Z(Z_X(s)) \subseteq Z_X(s)$ so the only non-trivial statement is that s is actually in $Z_X(s)^\circ = Y$. But, note that since s is semisimple, we have $s \in T(\mathbb{C})$ for T a maximal torus of X. Hence $s \in T(\mathbb{C}) \subset Y$ and so $s \in Y$ and thus $s \in Z(Y)$. Therefore, $Z_X(Z(Y)) \subseteq Z_X(s)$ and thus $Z_X(Z(Y))^\circ \subseteq Z_X(s)^\circ = Y$.

To finish the proof, it suffices to show that $N_X(Y)/Z_X(Z(Y))^\circ$ is finite. But, since $Z_X(Z(Y))^\circ$ is finite index in $Z_X(Z(Y))$ it suffices to show that $N_X(Y)/Z_X(Z(Y))$ is finite. Note though that $N_X(Y) \subseteq N_X(Z(Y))$ since $Z(Y)$ is a characteristic subgroup of Y. Thus, we get an inclusion

$$N_X(Y)/Z_X(Z(Y)) \hookrightarrow N_X(Z(Y))/Z_X(Z(Y)) \tag{52}$$

and thus it suffices to show this latter group is finite. Of course, this is equivalent to showing that $N_X(Z(Y))^\circ$ and $Z_X(Z(Y))^\circ$ coincide. Since $Z(Y)$ is multiplicative (since Y is reductive by [Hum11, §2.2]) this claim follows from [Hum12, Corollary, §16.3]. □

The second lemma is the following:

Lemma I.3.2. *Let F be a field of characteristic 0. Let X be reductive group over $\overline{F}$ and let $\mathcal{S}$ be a splitting of X. Then, given a finite Galois extension F'/F and a homomorphism $\xi : \mathrm{Gal}(F'/F) \to \mathrm{Out}(X)$, there exists a unique quasi-split group H over F such that there is an isomorphism $\widehat{H} \xrightarrow{\approx} X$ equivariant (up to inner automorphisms).*

Proof. Let Ψ be the based root datum associated to the triple (X, B, T) and let (X', B', T') be the dual triple with associated root datum $\Psi^\vee$. Let X'_0 be the unique split model of X' over F. Note then that we have natural isomorphisms of (constant) group (schemes)

$$\mathrm{Out}((X'_0)_{\overline{F}}) \cong \mathrm{Out}(X') \cong \mathrm{Aut}(\Psi^\vee) \cong \mathrm{Aut}(\Psi) \cong \mathrm{Out}(X) \tag{53}$$

Note then associated to ξ is a homomorphism $\xi^\vee : \mathrm{Gal}(F'/F) \to \mathrm{Out}((X'_0)_{\overline{F}})$. Then, by Proposition IV.4.5 we get a unique associated quasi-split inner form H of X'_0. Moreover, it's clear from construction that the natural map $\Gamma_F \to \mathrm{Out}(H_{\overline{F}})$ coincides with $\xi^\vee$. It is then not hard to see that we have a natural isomorphism $\overline{H} \xrightarrow{\approx} X$ as desired. □

We now return to the proof of Proposition I.2.15:

Proof. (Proposition I.2.15) We first define a map $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$. Pick a representative $(H, s, {}^L\eta, \psi^H)$ of $[H, s, {}^L\eta, \psi^H] \in \mathcal{EP}_F(G)$. We then get a parameter ψ of G given by $\psi^H \circ {}^L\eta$.

Now, by definition of endoscopic triple we have that $w \mapsto s^{-1}w(s)$ is a (locally) trivial 1-cocycle of W_F with values in $Z(\widehat{G})$ and this induces a (locally) trivial 1-cocycle of $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ via the projection $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F$. But then we have for all $w \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$

$$s^{-1}\psi^H(w)s\psi^H(w)^{-1} = s^{-1}w(s) \tag{54}$$

so that $\eta(s) \in S_\psi$. Conversely, pick an equivalence class $[\psi, \overline{s}] \in \mathcal{SP}_F(G)$ and pick a representative $(\psi, \overline{s})$. Let $s \in S_\psi$ be a lift of $\overline{s}$. Define $\widehat{H} :=$

$Z_{\widehat{G}}(s)^0$ and define η to be the natural embedding $\widehat{H} \hookrightarrow \widehat{G}$. Now, for any $g \in \mathrm{im}(\psi) \subset {}^L G$, the map $\mathrm{Int}(g) : \widehat{G} \to \widehat{G}$ stabilizes $\widehat{H}$ and hence gives a continuous homomorphism

$$\overline{\psi} : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Out}(\widehat{H}). \tag{55}$$

given by sending an element $(w, x) \in \mathcal{L}_F$ to the image of $\mathrm{Int}(\psi(w,x))_{|\widehat{H}}$ under the map $\mathrm{Aut}(\widehat{H}) \to \mathrm{Out}(\widehat{H})$. To see the continuity note that the map $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is definitionally continuous. The map ${}^L G \to \mathrm{Aut}({}^L G)$ is also clearly continuous. The map $\mathrm{Aut}({}^L G) \to \mathrm{Out}(\widehat{H})$ is continuous as one can clearly reduce to the split case in which case it reduces to checking the continuity of the map $\mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{H})$ but this is clear since this map of groups can be promoted to a functor of the associated group schemes. We claim that $\overline{\psi}$ has finite image. To see this note that it suffices to show that the image of $N_{{}^L G}(\widehat{H}) \to \mathrm{Out}(\widehat{H})$ has finite image. Note though that there is a finite extension E/F such that G_E is split so that ${}^L(G_E)$ is merely $\widehat{G} \times \Gamma_E$. Since ${}^L(G_E)$ is finite index in ${}^L G$ it's not hard to see that we can reduce to the case when G is split. The claim then immediately follows from Lemma I.3.1. Now note that any continuous finite quotient of $\mathcal{L}_F$ is of the form $\mathrm{Gal}(F'/F)$ for some finite extension F'/F. Indeed, evidently $\mathrm{SL}_2(\mathbb{C})$ has no non-trivial finite continuous quotients. Thus, it suffices to prove the claim for $\mathcal{L}_F$. Now, if K denotes the kernel of $\mathcal{L}_F \to W_F$ then K is a connected pro-reductive complex group. Thus, K also has no non-trivial finite continuous quotients. Thus, we've reduced the claim to W_F for which the claim is obvious. Thus, we have associated to (s, ψ) a homomorphism $\overline{\psi} : \mathrm{Gal}(F'/F) \to \mathrm{Out}(\widehat{H})$ which, by Lemma I.3.2, allows us to find a quasi-split group H over F whose dual group is naturally isomorphic to $\widehat{H}$ equivariant for the Γ_F actions on both sides.

We now claim that that (H, s, η) is an endoscopic datum for G. It remains to check that the conjugacy class of η is Γ_F-invariant and that the image of $s \in H^1(F, Z(\widehat{G}))$ is (locally) trivial. For the first check, we pick $w \in \Gamma_F$ and need to show that the constructed action of w on $\widehat{H}$ differs from the action of w on $\widehat{G}$ by an inner automorphism of $\widehat{G}$. In other words we need to show that for all $\sigma \in \Gamma_F$ that there exists some $g_\sigma \in \widehat{G}$ such that

$$\sigma_{\widehat{G}} \circ \eta \circ \sigma_{\widehat{H}}^{-1} = \mathrm{Int}(g_\sigma) \circ \eta \tag{56}$$

This is true by construction. For the second property, we note that the image of s in $H^1(F, Z(\widehat{G}))$ is definitionally given by $w \mapsto s^{-1}w(s)$ for $w \in \Gamma_F$. Since Γ_F acts on $\widehat{H}$, and thus $Z(\widehat{G}) \subseteq \widehat{H}$, through $\mathrm{Gal}(F'/F)$ we see that this cocycle is induced from a cocycle in $H^1(\mathrm{Gal}(F'/F), Z(\widehat{G}))$. Now we observe that for any lift $w' \in \mathcal{L}_F \times SL_2(\mathbb{C})$ of w, we have

$$s^{-1}\psi(w')s\psi(w')^{-1} = s^{-1}w(s). \tag{57}$$

Since $s \in S_\psi$, this gives the desired result.

By our assumption that G^{der} is simply connected, we can extend η to a map ${}^L\eta : {}^LH \to {}^LG$. Then we need to check that the parameter ψ factors through ${}^L\eta$. We shall follow techniques discussed in unpublished notes of Kottwitz. Let us begin by defining the subgroup $\mathcal{H}$ of ${}^LG$ as the set of elements $x \in {}^LG$ such that there exists an element $y \in {}^LH$ such that the equality

$$\mathrm{Int}(x) \circ {}^L\eta = {}^L\eta \circ \mathrm{Int}(y), \tag{58}$$

holds. Note that $\mathcal{H}$ depends only on ${}^L\eta \mid_{\widehat{H}}$ and, in particular, only on the endoscopic triple (H, s, η). We then have the following observation of Kottwitz:

Lemma I.3.3. *The set $\mathcal{H}$ is a subgroup of ${}^LG$ which is a split extension of W_F by $\widehat{H}$.*

Proof. The proof is due to unpublished work of Kottwitz.

There exists a finite extension K/F such that the action of Γ_F on $\widehat{H}$ and $\widehat{G}$ factors through Γ_K. Now pick $\sigma \in \mathrm{Gal}(K/F)$ and $w \in W_F$ such that w projects to $\sigma \in \mathrm{Gal}(K/F)$. Then $(1, w) \in {}^LH$ acts on $\widehat{H}$ by σ. By definition, there exists a $g_\sigma \in \widehat{G}$ such that $\mathrm{Int}(g_\sigma) \circ \eta = \sigma \cdot \eta$. Then

$$\eta \circ (1, w) = \mathrm{Int}(\sigma(g_\sigma), w)) \circ \eta, \tag{59}$$

which implies $\mathcal{H}$ surjects onto W_F.

Now the kernel of $\mathcal{H} \to W_F$ consists of $x \in \widehat{G}$ such that there exists $y \in \widehat{H}$ and $\mathrm{Int}(x) \circ \eta = \eta \circ \mathrm{Int}(y)$. Clearly $\eta(\widehat{H})$ is contained in this set. Conversely, we have that $\mathrm{Int}(x^{-1}\eta(y))$ acts trivially on $\widehat{H}$. In particular, $x^{-1}\eta(y)$ must centralize a maximal torus $\widehat{T}_H$ of $\eta(\widehat{H})$. Then $\widehat{T}_H$ is maximal in $\widehat{G}$ as well so $x^{-1}\eta(y) \in \widehat{T}_H \subset \eta(\widehat{H})$. Hence $x \in \eta(\widehat{H})$.

We now prove that the extension

$$1 \to \eta(\widehat{H}) \to \mathcal{H} \to W_F \to 1 \tag{60}$$

is split. We proceed as follows. Let $\widehat{T} \subset \widehat{B}$ be maximal torus and Borel of $\widehat{H}$ and let $\mathcal{T}$ be the subgroup of $\mathcal{H}$ of elements preserving the pair $(\eta(\widehat{T}), \eta(\widehat{B}))$. Then $\mathcal{T}$ is an extension of W_F by $\eta(\widehat{T})$.

Then [Lan79, Lemma 4] says that if there exists a field K that is a finite Galois extension of F such that the action of W_F on $\widehat{T}$ factors through $\mathrm{Gal}(K/F)$, then $\mathcal{T}$ is split. Since this is the case, $\mathcal{T}$ is split so we can take a splitting $c : W_F \to \mathcal{T}$. Then this is also a splitting of $\mathcal{H}$.

□

We then observe that for any ${}^L\eta$, we have ${}^L\eta({}^LH) \subset \mathcal{H}$. In particular, ${}^L\eta$ gives a map of extensions of W_F by $\eta(\widehat{H})$ and hence is an isomorphism onto $\mathcal{H}$.

Thus, to show that ψ factors through ${}^L\eta$, we need only show that $\mathrm{im}(\psi) \subset \mathcal{H}$. We need to show that for each $x \in \mathrm{im}(\psi)$, there exists $y \in {}^LH$ such that the projections of x and y to W_F agree and

$$\mathrm{Int}(x) \circ \eta = \eta \circ \mathrm{Int}(y), \tag{61}$$

on $\widehat{H}$. First pick $w \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ and consider $\psi(w)$. Then we check that there exists an element $y \in {}^LH$ such that $\mathrm{Int}(\psi(w)) \circ \eta = \eta \circ \mathrm{Int}(y)$. But indeed this follows immediately from the fact that the L-action of the projection $\overline{w} \in W_F$ on $\widehat{H} \subset {}^LH$ differs from that of $\mathrm{Int}(\psi(w))$ by an element of $\mathrm{Inn}(\widehat{H})$. We then define a parameter ψ^H such that ${}^L\eta \circ \psi^H = \psi$.

We now show the map we have constructed is well-defined. First, one can also easily show that choosing a different lift of $\overline{s}$ gives an isomorphic endoscopic datum. Next, suppose that $(\psi_1, \overline{s_1})$ is equivalent to $(\psi_2, \overline{s_2})$ by some $g \in \widehat{G}$ satisfying $w \mapsto g\psi_1(w)g^{-1}\psi(w)_2^{-1}$ is a (locally) trivial cocycle of $\mathcal{L}_F$ valued in $Z(\widehat{G})$. Then by assumption $g\overline{s_1}g^{-1}$ is conjugate by some $s \in S_{\psi_2}$ to $\overline{s_2}$ and so the groups $\widehat{H}_1$ and $\widehat{H}_2$ are conjugate in $\widehat{G}$ by sg. Moreover, it is easy to check that the map $\mathrm{Int}(sg) : \widehat{H}_1 \to \widehat{H}_2$ will preserve the actions of Γ_F up to an inner automorphism of $\widehat{H}_2$ and hence descends to an isomorphism $\alpha : H_2 \to H_1$ defined over F. The map α then gives an isomorphism of the endoscopic data (H_1, s_1, η_1) and (H_2, s_2, η_2) and ${}^L\mathrm{Int}(sg) \circ \psi_1^H$ is $Z(G)$-equivalent to ψ_2^H. This shows the map is well-defined.

To conclude the proof, we must show that the maps $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$ and $\mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ that we have constructed are inverses of each other. It is clear that the composition $\mathcal{SP}_F(G) \to \mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$ is the identity. Indeed, the first map sends $[\overline{s}, \psi]$ to an element of $\mathcal{EP}_F(G)$ of the form $[H, s, {}^L\eta, \psi^H]$ where s is a lift of $\overline{s}$ to S_ψ and ${}^L\eta \circ \psi^H = \psi$. The second map then takes $[H, s, {}^L\eta, \psi^H]$ to $[\overline{\eta(s)}, {}^L\eta \circ \psi^H]$. But, by definition $\overline{\eta(s)} = \overline{s}$ and ${}^L\eta \circ \psi^H = \psi$ from where the conclusion follows.

We now show that the composition $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ is the identity. Take a representative $(H, s, {}^L\eta, \psi^H)$ of $[H, s, {}^L\eta, \psi^H] \in \mathcal{EP}_F(G)$. Then we want to show that this is equivalent to the tuple $(H', s', {}^L\eta', \psi^{H'})$ that we get from applying the composition $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ to $(H, s, {}^L\eta, \psi^H)$. Note that, up to equivalence, we can assume that $s' = s$ and so we have a map of complex Lie groups ${\eta'}^{-1} \circ \eta : \widehat{H} \to \widehat{H'}$.

We claim this map is equivariant for each $w \in \Gamma_F$ up to conjugation by some $h \in \widehat{H}$. There exists some finite extension E/F such that the actions of Γ_F on both groups factor through $\mathrm{Gal}(E/F)$ hence we need only prove the claim for $w \in \mathrm{Gal}(E/F)$. Pick a lift $w' \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ of w, the action of w on each group differs by an inner automorphism from the action of conjugation by $\psi^H(w')$ or $\psi^{H'}(w')$ respectively. So then we have (up to conjugation which we denote by $\sim$) for $h \in \widehat{H}$:

$$(w \cdot (\eta'^{-1} \circ \eta))(h) = w(\eta'^{-1}\eta(w^{-1}(h))) \tag{62}$$
$$\sim \operatorname{Int}(\psi^{H'}(w'))(\eta'^{-1}\eta(\operatorname{Int}(\psi^{H}(w')^{-1})(h)) \tag{63}$$
$$= (\eta'^{-1} \circ \operatorname{Int}(\psi(w')) \circ \operatorname{Int}(\psi(w')^{-1}) \circ \eta)(h) \tag{64}$$
$$= (\eta' \circ \eta)(h). \tag{65}$$

This proves the claim and implies that the isomorphism descends to an isomorphism $\alpha : H' \to H$ defined over F. This satisfies $\widehat{\alpha}(s) = s' \mod Z(\widehat{G})$ and hence gives the desired isomorphism of endoscopic data. Moreover, it is clear that we have an equivalence $(H, s, {}^L\eta, \psi^H), (H', s', {}^L\eta', \psi^{H'})$.

We now check that the bijection restricts to give a bijection

$$\mathcal{EP}_F^{\mathrm{ell}}(G) \to \mathcal{SP}_F^{\mathrm{ell}}(G). \tag{66}$$

We need to check that if $[\psi, \overline{s}] \in \mathcal{SP}_F(G)^{\mathrm{ell}}$, then the tuple $(H, s, {}^L\eta, \psi^H)$ we construct from $(\psi, \overline{s})$ satisfies that (H, s, η) is elliptic. But we have $\eta((Z(\widehat{H})^{\Gamma_F})^0) \subset \eta(C^0_{\psi^H}) \subset C^0_\psi \subset Z(\widehat{G})$ as desired. Note that the last equality holds by [Kot84b, lemma 10.3.1]. □

I.4 Proof of I.2.17

We now prove our main result on relevancy of global endoscopy. We need to construct a $(\mathbf{G}, \mathbf{H})$-regular $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $\gamma_{\mathbf{H}}$ transfers to some elliptic $\gamma \in \mathbf{G}(F)$. To do so, we first need the following proposition.

Proposition I.4.1 ([Kot90, pg 188])**.** *$\mathbf{G}$ be a group over a totally number field F. Let $(\mathbf{H}, s, \eta)$ be an endoscopic datum of $\mathbf{G}$ such that $(\mathbf{H}_v, s, \eta)$ is elliptic for all infinite places v of F. Let $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ be a $(\mathbf{G}, \mathbf{H})$-regular semisimple element such that $\gamma_{\mathbf{H}}$ transfers to an element of $\mathbf{G}(F_v)$ for each place v of F and $\gamma_{\mathbf{H}}$ is elliptic as an element of $\mathbf{H}(F_v)$ for all infinite places v of F. Then in fact, $\gamma_{\mathbf{H}}$ transfers to a semisimple $\gamma \in \mathbf{G}(F)$.*

Let us note that it suffices to consider the case when $F = \mathbb{Q}$. Indeed, set $\mathbf{G}' := \mathsf{Res}_{F/\mathbb{Q}}\mathbf{G}$ and set $(\mathbf{H}', s', \eta')$ to be so that $\mathbf{H}' = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{H}$, the element $s' := (s, ..., s) \subset \widehat{\mathbf{H}'} = \widehat{\mathbf{H}}^m$ (where $m := [F : \mathbb{Q}]$), and η' is the map $\widehat{\mathbf{H}'} \to \widehat{\mathbf{G}'}$ given by

$$\eta'(h_1, \dots, h_m) := (\eta(h_1), \dots, \eta(h_1), \dots, \eta(h_m)) \tag{67}$$

Then, if we let $\gamma_{\mathbf{H}'}$ be equal to $\gamma_{\mathbf{H}}$ as an element of $\mathbf{H}'(\mathbb{Q}) = \mathbf{H}(F)$ we get the desired result.

Before we begin the proof in earnest, we record here a general fact:

Lemma I.4.2. *Let X be a reductive group over a field F. Then, there is a short exact sequence of Γ_F-modules*

$$1 \to K \to Z(\widehat{X})^\circ \to \widehat{Z(X)^\circ} \to 1 \tag{68}$$

where K is some finite Γ_F-module. If F is a local field, this in turn induces a natural isogeny of abelian groups

$$(Z(\widehat{X})^\circ)^{\Gamma_F} \to (\widehat{Z(X)^\circ})^{\Gamma_F} \tag{69}$$

Proof. Let us begin by noting that we have a short exact sequence of connected reductive F-groups

$$1 \to Z(X)^\circ \to X \to Q \to 1 \tag{70}$$

where $Q := X/Z(X)^\circ$ is semisimple. We then get a short exact sequence of Γ_F-modules

$$1 \to Z(\widehat{Q}) \to Z(\widehat{X}) \to \widehat{Z(X)^\circ} \to 1 \tag{71}$$

Note that since Q is semisimple, $Z(\widehat{Q})$ is finite (e.g. [Kot84b, (1.8.4)]) from where the first part of the proposition follows.

Let us now consider the associated long exact sequence of Γ_F-modules

$$1 \to Z(\widehat{Q})^{\Gamma_F} \to (Z(\widehat{X})^\circ)^{\Gamma_F} \to (\widehat{Z(X)^\circ})^{\Gamma_F} \to H^1(F, Z(\widehat{Q})) \tag{72}$$

We are then done by observing that since F is a local field that $H^1(F, Z(\widehat{Q}))$ is finite. $\square$

Proof. (Proposition I.4.1) By assumption there exists a $\gamma \in \mathbf{G}(\mathbb{A})$ such that $\gamma_{\mathbf{H}}$ transfers to γ. Let $\psi : \mathbf{G}^* \to \mathbf{G}$ be a quasisplit inner twist of $\mathbf{G}$. By [Kot82, Theorem 4.1], $\gamma_{\mathbf{H}}$ transfers to some $\gamma^* \in \mathbf{G}^*(\mathbb{Q})$.

Now, as in [Kot86b, §6], the elements γ^*, γ determine an element $\mathrm{obs}(\gamma) \in \mathfrak{K}(I_{\gamma^*}/\mathbb{Q})^D$ such that γ is conjugate in $\mathbf{G}(\mathbb{A})$ to an element of $\mathbf{G}(\mathbb{Q})$ if and only if $\mathrm{obs}(\gamma)$ is trivial.

Lemma I.4.3. *The element $\gamma^* \in \mathbf{G}(\mathbb{R})$ is $\mathbb{R}$-elliptic.*

Proof. Since $\gamma_{\mathbf{H}}$ is $(\mathbf{G}, \mathbf{H})$-regular and elliptic in $\mathbf{H}(\mathbb{R})$, it follows that γ^* is elliptic in $\mathbf{G}^*(\mathbb{R})$. Indeed, recall first that since $\mathbf{H}$ is an endoscopic group of G that $Z(\mathbf{G}) \subseteq Z(\mathbf{H})$ as $\mathbb{Q}$-groups (e.g. see the second to last paragraph of [Shi10, Page 5]). Note then that since $\gamma_{\mathbf{H}}$ is $(\mathbf{G}, \mathbf{H})$-regular that I_γ and I_{γ^*} are inner forms (e.g. see [Kot86b, §3]). Thus,

$$Z(\mathbf{G}) \subseteq Z(\mathbf{H}) \subseteq Z(I_\gamma) = Z(I_{\gamma^*}) \tag{73}$$

holds and thus

$$Z(\mathbf{G}_{\mathbb{R}}) \subseteq Z(\mathbf{H}_{\mathbb{R}}) \subseteq Z(I_{\gamma,\mathbb{R}}) = Z(I_{\gamma^*,\mathbb{R}}) \tag{74}$$

holds by base change.

To show that γ^* is elliptic we need to show that $Z(I_{\gamma^*,\mathbb{R}})^\circ/Z(\mathbf{G}_\mathbb{R})^\circ$ is $\mathbb{R}$-anisotropic. By assumption we have that $Z(I_{\gamma,\mathbb{R}})^\circ/Z(H_\mathbb{R})^\circ$ is $\mathbb{R}$-anisotropic. Since (H,s,η) is $\mathbb{R}$-elliptic we have that $Z(\mathbf{H}_\mathbb{R})^\circ_s = Z(\mathbf{G}_\mathbb{R})^\circ_s$ (e.g. see the second to last paragraph of [Shi10, Page 5]), which implies the desired consequence. □

Lemma I.4.4. *The containment $(Z(\widehat{I_{\gamma^*}})^{\Gamma_\infty})^\circ \subset Z(\widehat{\mathbf{G}})$ holds.*

Proof. Begin by noting that

$$Z(\widehat{I_{\gamma^*}})^{\Gamma_\infty} = Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty} \tag{75}$$

Now, by assumption we have that $T := Z(I_{\gamma^*,\mathbb{R}})^\circ$ is an elliptic torus in $\mathbf{G}_\mathbb{R}$. Then, by lemma IV.1.37 implies that $(\widehat{T}/Z(\widehat{\mathbf{G}}))^{\Gamma_\infty}$ is finite (note that $Z(\widehat{\mathbf{G}}) = Z(\widehat{\mathbf{G}_\mathbb{R}})$ so we ignore the difference). Thus, *a foritiori*, we know that $\widehat{T}^{\Gamma_\infty}/Z(\widehat{\mathbf{G}})^{\Gamma_\infty}$ is finite. In particular, since $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $Z(\widehat{\mathbf{G}})^{\Gamma_\infty}$, we have that $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_\infty}$.

Now, note that we're trying to show that $((Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ \subseteq Z(\widehat{\mathbf{G}})$ so it suffices to show that $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ = (Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$. Note that evidently $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is contained in $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ$, and since the latter is connected it suffices to show that the former is finite index in the latter.

Now, we know that $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_\infty}$. Note though that by Lemma I.4.2 we have an isogeny of abelian groups

$$(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ)^{\Gamma_\infty} \to ((Z(I_{\gamma^*,\mathbb{R}})^\circ)\widehat{\ })^{\Gamma_\infty} =: \widehat{T}^{\Gamma_\infty} \tag{76}$$

which is equivariant for the inclusions of $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ on both sides. In particular, since $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_F}$ it's also finite index in $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$.

Note then that we have the exact sequence of Γ_∞-modules

$$1 \to Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ \to Z(\widehat{I_{\gamma^*,\mathbb{R}}}) \to \pi_0(Z(\widehat{I_{\gamma^*,\mathbb{R}}})) \to 1 \tag{77}$$

which gives us the exact sequence

$$1 \to (Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ)^{\Gamma_\infty} \to Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty} \to \pi_0(Z(\widehat{I_{\gamma^*,\mathbb{R}}}))^{\Gamma_\infty} \tag{78}$$

which shows that, since $\pi_0(Z(\widehat{I_{\gamma^*,\mathbb{R}}}))$ is finite, that $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$ is finite index in $Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty}$. Since $(Z(\widehat{G})^{\Gamma_\infty})^\circ$ is finite index in $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$ it follows that it's also finite index in $Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty}$. It follows that $(Z(\widehat{G})^{\Gamma_\infty})^\circ$ must be finite index in $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ$ from where the conclusion follows. □

Now, the action of Γ on $Z(\widehat{I_{\gamma^*}})$ factors through some finite quotient Γ_K let σ be the nontrivial element of $\Gamma_\mathbb{R}$. This gives a conjugacy class $\{\sigma\} \subset \Gamma_K$.

Then by Cebotarev Density, we can find some finite place v of $\mathbb{Q}$ such that the conjugacy class of Frob_v equals $\{\sigma\}$. In particular, for such a v, we have

$$(Z(\widehat{I_{\gamma^*}})^{\Gamma_v})^0 \subset Z(\widehat{I_{\gamma^*}})^{\Gamma_\infty} \subset Z(\widehat{\mathbf{G}}). \tag{79}$$

Now, recall that the set of $\mathbf{G}(\mathbb{Q}_v)$ conjugacy classes in the stable conjugacy class of γ^* is in bijection with $\ker[H^1(\mathbb{Q}_v, I_{\gamma^*}) \to H^1(\mathbb{Q}_v, \mathbf{G})]$. Then by the Kottwitz isomorphism we have the bijection

$$\ker[H^1(\mathbb{Q}_v, I_{\gamma^*}) \to H^1(\mathbb{Q}_v, \mathbf{G})] \cong \ker[\pi_0(Z(\widehat{I_{\gamma^*}})^{\Gamma_v})^D \to \pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v})^D]. \tag{80}$$

Now, $\mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v)$ equals the image of $Z(\widehat{I_{\gamma^*}})^\Gamma_v$ under the map

$$Z(\widehat{I_{\gamma^*}})^\Gamma_v \to [Z(\widehat{I_{\gamma^*}})/Z(\widehat{\mathbf{G}})]^{\Gamma_v}. \tag{81}$$

Since the kernel of this map is $Z(\mathbf{G})^{\Gamma_v}$ and we have shown that in our case

$$(Z(\widehat{I_{\gamma^*}})^\Gamma)^0 \subset Z(\widehat{\mathbf{G}}), \tag{82}$$

it follows that in fact, the map

$$Z(\widehat{I_{\gamma^*}})^\Gamma_v \to [Z(\widehat{I_{\gamma^*}})/Z(\widehat{\mathbf{G}})]^{\Gamma_v} \tag{83}$$

factors through $\pi_0(Z(\widehat{I_{\gamma^*}})^{\Gamma_v})$ and hence, we have an exact sequence

$$\pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v}) \to \pi_0(Z(\widehat{I_{\gamma^*}})^{\Gamma_v} \to \mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v) \to 1. \tag{84}$$

Dualizing gives

$$\mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v)^D = \ker[\pi_0(Z(\widehat{I_{\gamma^*}})^{\Gamma_v})^D \to \pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v})^D], \tag{85}$$

and so in conclusion, we have a bijection

$$\ker[H^1(\mathbb{Q}_v, I_{\gamma^*}) \to H^1(\mathbb{Q}_v, \mathbf{G})] \twoheadrightarrow \mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v)^D. \tag{86}$$

By definition, we have a surjection

$$\mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v)^D \twoheadrightarrow \mathfrak{K}(I_{\gamma^*}/\mathbb{Q})^D. \tag{87}$$

Finally, we observe that $\mathfrak{K}(I_{\gamma^*}/\mathbb{Q}_v) \cong \mathfrak{K}(I_\gamma/\mathbb{Q}_v)$ so that we in fact have a surjection

$$\ker[H^1(\mathbb{Q}_v, I_\gamma) \to H^1(\mathbb{Q}_v, \mathbf{G})] \twoheadrightarrow \mathfrak{K}(I_{\gamma^*}/\mathbb{Q})^D. \tag{88}$$

In particular, it follows that we can modify γ at the place v by some stable conjugate such that $\mathrm{obs}(\gamma)$ vanishes. This then implies the desired result. □

We now return to the proof of I.2.17. By I.4.1, we just need to find a semisimple $(\mathbf{G},\mathbf{H})$-regular $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ that transfers to each $\mathbf{G}(F_v)$ and is elliptic at each real place.

We now reduce the question of transferring $\gamma_{\mathbf{H}}$ to that of transferring a torus T of $\mathbf{H}$. More precisely, we record the following lemma

Lemma I.4.5. *Let (H, s, η) be an endoscopic group for G such that H and G are defined over a local field F. Suppose $T \subset H$ is a maximal torus defined over F and that T transfers to G in the sense of [Shi10] after remark 2.6. Then for any semisimple $\gamma \in \mathbf{T}(F)$, we have that γ transfers to $G(F)$ in the sense of [Shi10, §2.3].*

Proof. This is clear from definition. □

Hence, to prove I.2.17, it suffices to find a maximal torus $\mathbf{T} \subset \mathbf{H}$ defined over F that transfers to $\mathbf{G}$ since the $(\mathbf{G},\mathbf{H})$-regular elements are dense in $\mathbf{T}$. By IV.1.12, there exists a $\mathbf{T}$ defined over F and such that for each place v of F that $\mathbf{G}_v$ is not quasisplit, we have $\mathbf{T}_v$ is elliptic. In the quasisplit cases, it is clear that $\mathbf{T}_v$ transfers. Hence it suffices to show that if $(\mathbf{H}_v, s, {}^L\eta_v, \psi_v^{\mathbf{H}}, \mathbf{T}_v)$ is such that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum, $\psi_v^{\mathbf{H}}$ is an A-parameter of $\mathbf{H}_v$ such that ${}^L\eta \circ \psi_v^{\mathbf{H}}$ is a relevant parameter of $\mathbf{G}_v$, and $\mathbf{T}_v$ is an elliptic maximal torus of $\mathbf{H}_v$ defined over F_v, then $\mathbf{T}_v$ transfers to $\mathbf{G}_v$.

Now consider the torus $\eta_v((Z(\widehat{\mathbf{H}}_v)^{\Gamma_{F_v}})^\circ) \subset \widehat{\mathbf{G}}_v$. Then the centralizer of this torus in ${}^L\mathbf{G}_v$ surjects onto W_{F_v} since it contains ${}^L\eta({}^L\mathbf{H}_v)$. In particular, we have that $Z_{{}^L\mathbf{G}_v}(\eta_v((Z(\widehat{\mathbf{H}}_v)^{\Gamma_{F_v}})^\circ))$ is a Levi subgroup of ${}^L\mathbf{G}_v$ by [Bor79, Lemma 3.5]. To simplify notation, we denote this subgroup $\mathcal{M}$. By assumption, since clearly ${}^L\eta_v$ factors through $\mathcal{M}$, we have that $\mathcal{M}$ is relevant. Hence $\mathcal{M}$ in conjugate by an element of $\widehat{\mathbf{G}}_v$ to a subgroup ${}^LM \subset {}^L\mathbf{G}_v$ such that $M \subset \mathbf{G}_v$ is a standard Levi subgroup. Since we are only concerned with the endoscopic datum $(\mathbf{H}_v, s, \eta_v)$ up to isomorphism, we can replace it with any isomorphic datum $(\mathbf{H}_v, s, \eta_v \circ \mathrm{Int}(g))$. In particular, we can and do assume without loss of generality that $\mathcal{M} = {}^LM$.

We claim that $(\mathbf{H}_v, s, \eta_v)$ is an elliptic endoscopic datum for M. We first check that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum for M. To see that the conjugacy class of η_v is Γ_{F_v}-invariant, we note that ${}^L\eta({}^L\mathbf{H}_v) \subset \mathcal{M}$. Since W_{F_v} and Γ_{F_v} act through some finite quotient $\mathrm{Gal}(K/F_v)$ on $\widehat{\mathbf{H}}_v$ and $\widehat{\mathbf{G}}_v$, it suffices to show that the conjugacy class of η is invariant under the action of some arbitrary $\sigma \in \mathrm{Gal}(K/F_v)$. Let $w \in W_{F_v}$ be a lift of σ. Then ${}^L\eta(1, w) = (m, w) \in {}^LM$ and we have

$$\begin{aligned}
\sigma \cdot \eta &= \sigma_{\widehat{\mathbf{G}}_v} \circ \eta \circ \sigma_{\widehat{\mathbf{H}}_v}^{-1} && (89)\\
&= \mathrm{Int}((1, w)) \circ \eta \circ \mathrm{Int}((1, w^{-1})) && (90)\\
&= \mathrm{Int}((1, w)(w^{-1}(m^{-1}), w^{-1})) \circ \eta && (91)\\
&= \mathrm{Int}(m^{-1}) \circ \eta, && (92)
\end{aligned}$$

as desired. The only remaining check to show that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum is that the image of s in $H^1(F_v, Z(\widehat{M})^{\Gamma_{F_v}})$ is trivial, but this follows immediately from the functoriality of these cohomology groups. Finally, to prove that the datum is elliptic, we observe that by assumption, $\eta_v((Z(\widehat{\mathbf{H}_v})^{\Gamma_{F_v}})^\circ) \subset Z(\widehat{M})$.

Now, we transfer $\mathbf{T}_v$ to M^* and observe that since the endoscopic datum is elliptic, $\mathbf{T}_v$ must be elliptic in M^*. In particular, it follows that $\mathbf{T}_v$ transfers to M and therefore $\mathbf{G}_v$. This completes the proof.

I.5 No relevant global endoscopy

Our goal in this section is to discuss the case where a group $\mathbf{G}$ possesses no relevant endoscopic groups other than the trivial one.

Namely, let us make the following definition:

Definition I.5.1. *Let $\mathbf{G}$ be a reductive group over a number field F. We say that $\mathbf{G}$ has* no relevant global endoscopy *if $\mathcal{RE}(\mathbf{G})$ consists (up to equivalence) only of the trivial endoscopic triple $(\mathbf{G}, e, \mathrm{id})$. We say that $\mathbf{G}$ has* no relevant global elliptic endoscopy *if $\mathcal{RE}^{\mathrm{ell}}(\mathbf{G})$ consists (up to equivalence) only of the trivial endoscopic triple $(\mathbf{G}, e, \mathrm{id})$.*

We make the following useful observation:

Lemma I.5.2. *Let $\mathbf{G}$ be a reductive group over a number field F. Then, $\mathbf{G}$ has no relevant global endoscopy if and only if for all semi-simple $\gamma \in \mathbf{G}(F)$ we have that $\mathfrak{K}(I_\gamma/F) = 0$. Similarly, $\mathbf{G}$ has no relevant global elliptic endoscopy if for all semi-simple and elliptic $\gamma \in \mathbf{G}(F)$ we have that $\mathfrak{K}(I_\gamma/F) = 0$.*

Proof. Suppose first that $\mathbf{G}$ has no relevant global endoscopy. Pick $(\gamma, \kappa) \in \mathcal{SS}_F(\mathbf{G})$. Note then that by Proposition I.2.6, we get an element $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \in \mathcal{EQ}_F(\mathbf{G})$ associated to (γ, κ). By assumption, we then know that $(\mathbf{H}, s, \eta) \sim (\mathbf{G}, e, \mathrm{id})$ and so in particular, $\eta(s) \in Z(\widehat{G})$, which implies κ is trivial.

Conversely, suppose that $\mathfrak{K}(I_\gamma/F)$ is trivial for all semi-simple $\gamma \in \mathbf{G}(F)$. Let $(\mathbf{H}, s, \eta)$ be an element of $\mathcal{RE}(\mathbf{G})$. Choose some semi-simple $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}})$ is an element of $\mathcal{EQ}_F(\mathbf{G})$. Note that by Proposition I.2.6 we get associated to this quadruple a pair $(\gamma, \kappa) \in \mathcal{SS}_F(\mathbf{G})$. By our assumption we have that $\kappa = 0$. Pick a transfer γ^* of γ to $\mathbf{G}^*(F)$. Then $(\mathbf{G}^*, e, \mathrm{id}, \gamma)$ is an element of $\mathcal{EQ}_F(\mathbf{G})$ which maps to $(\gamma, 0)$ under Proposition I.2.6. Thus, we deduce that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \sim (\mathbf{G}^*, e, \mathrm{id}, \gamma)$ as desired.

The elliptic version is similar. $\square$

We will be mostly interested in reductive groups $\mathbf{G}$ such that $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic and which satisfy the Hasse principle (i.e. that $\ker^1(F, \mathbf{G}) = 0$), in which case the condition of no relevant global (elliptic) endoscopy takes the following particularly simple form:

Proposition I.5.3. *Let F be a number field and $\mathbf{G}$ be a reductive group over F. Assume further that $\mathbf{G}^{\mathrm{ad}}$ is F-anistropic and satisfies the Hasse principle. Then, the following are equivalent:*

1. *$\mathbf{G}$ has no relevant global endoscopy.*
2. *$\mathbf{G}$ has no relevant global elliptic endoscopy.*
3. *For all maximal F-tori $T \subset \mathbf{G}$ one has that the containment $Z(\widehat{\mathbf{G}})^\Gamma \subseteq \widehat{T}^\Gamma$ is actually an equality.*

Proof. Let us begin by observing that 1. and 2. are equivalent simply because every semi-simple element of $\mathbf{G}(F)$ is elliptic. Thus, it suffices to prove the equivalence of 1. and 3.

Note that since $\mathbf{G}$ satisfies the Hasse principle, we have that $\ker^1(\Gamma, Z(\widehat{\mathbf{G}}))$ vanishes (e.g. see [Kot84b, Remark 4.4]). Thus, it's fairly easy to see that for any semi-simple γ in $\mathbf{G}(F)$ we have that

$$\mathfrak{K}(I_\gamma/F) = Z(\widehat{I_\gamma})^\Gamma / Z(\widehat{\mathbf{G}})^\Gamma \tag{93}$$

and thus the implication of 3. implies 1. follows immediately from Lemma IV.1.36. The implication that 1. implies 3. would follow quite simply if every maximal torus T in G were of the form I_γ for some semi-simple $\gamma \in \mathbf{G}(F)$. But, this follows immediately from Theorem IV.1.20.

□

I.6 An application to the representation theory of unitary groups

In this section, we derive some results on the representation theory of global unitary groups with no relevant global endoscopy. In particular, we show that the relevant elliptic A-parameters of such groups satisfy $\overline{S_\psi} = 1$. While one could prove this in enough cases using special assumptions to prove our main result, we prefer the present, more systematic, approach.

Let $F/\mathbb{Q}$ be a total real extension of number fields and E/F be a quadratic imaginary extension. Let n be an odd natural number and $(U_{E/F}(n), \omega)$ be an inner twist of $U_{E/F}(n)^*$ having no relevant endoscopy. Such a group exists by III.1.2.

In the course of our proof, we need to appeal to the bijection I.2.15 in the global case. To avoid making assumptions about the global Langlands group $\mathcal{L}_\mathbb{Q}$, we work with "automorphic A-parameters" in the sense of [KMSW14, §1.3.4]. This notion is originally due to Arthur [Art13a]. We note that an automorphic parameter yields at each place v of F, a localization ψ_v which is an A-parameter of $\mathbf{U}_v$ [KMSW14, §1.3.5]. Moreover, one can make sense of the groups C_ψ and S_ψ for such parameters [KMSW14, §1.3.4]. In particular, we note that the words elliptic and relevant make sense for automorphic parameters. Thus, a first step is to prove a version of I.2.15 for automorphic parameters.

Proposition I.6.1. *Let E/F be a quadratic extension of number fields. Let $\mathbf{U}$ be an inner form of $U_{E/F}(N)^*$. Let us make the following notational definitions*

- *Set $\mathcal{AEP}_F(\mathbf{U})$ to be the set of all quadruples $(H, s, {}^L\eta, \psi^{\mathfrak{e}})$ where $(H, s, {}^L\eta)$ is an extended endoscopic datum of $\mathbf{U}$ And $\psi^{\mathfrak{e}} = (\psi^n, \tilde{\psi}^{\mathfrak{e}}) \in \Psi(H, {}^L\eta)$ (as in [KMSW14, §1.3.6]).*
- *Set $\mathcal{ASP}_F(\mathbf{U})$ to be the set of all pairs $(\overline{s}, \psi)$ where $\psi = (\psi^n, \tilde{\psi}) \in \Psi(\mathbf{U}, \eta_{\chi_k})$ and $\overline{s} \in \overline{S_\psi}$.*

We then have a bijection $\mathcal{AEP}_F(\mathbf{U}) \to \mathcal{ASP}_F(\mathbf{U})$ given by

$$[H, s, {}^L\eta, \psi^H] \mapsto [{}^L\eta \circ \tilde{\psi}^{\mathfrak{e}}, \overline{\eta(s)}], \tag{94}$$

Moreover, this bijection is compatible via localization with the local version of I.2.15 using the localization map in [KMSW14, §1.3.5].

Proof. The bijection is constructed analogously to the proof of I.2.15. We first define the inverse map. Given $[\overline{s}, \psi] \in \mathcal{ASP}_F(\mathbf{U})$ we need to construct an element of $\mathcal{AEP}_F(\mathbf{U})$, In particular, $\mathcal{L}_\psi$ is an extension of W_F by a pro-reductive group just as $\mathcal{L}_F$ was. Since this was the key property of $\mathcal{L}_F$ that we used, we can construct the datum $(H, s, {}^L\eta)$ using a lift of $\overline{s}$ and $\tilde{\psi} : \mathcal{L}_\psi \to {}^L\mathbf{U}$ as in the proof of I.2.15. Then we can conclude as before that $\tilde{\psi}$ factors through the image of ${}^L\eta$ and hence gives rise to a parameter $\psi^{\mathfrak{e}}$ such that $\eta_\chi \circ {}^L\eta \circ \tilde{\psi}^{\mathfrak{e}} = \tilde{\psi}^n$ as desired. As in I.2.15 we conclude that this map is the desired inverse.

Now we prove compatibility with the local version of I.2.15. We need to show that if v is a place of F, then the bijection in I.2.15 identifies $[H_v, s_v, {}^L\eta_v, \psi_v^{\mathfrak{e}}]$ with $[\overline{s}, \psi_v]$. This follows from the commutative diagram after Proposition 1.3.3 in [KMSW14]. $\square$

I.6.1 The Triviality of $\overline{S_\psi}$

In this subsection, we prove that relevant elliptic parameters of the group $\mathbf{U} := U_{E/F}(n)$ satisfy $\overline{S_\psi} = 1$.

Proposition I.6.2. *Let ψ be a relevant elliptic automorphic A-parameter of $\mathbf{U}$ such that for some infinite place v_∞ of F, we have ψ_{v_∞} is elliptic. Then we have $\overline{S_\psi} = 1$.*

Proof. Suppose for contradiction that $\overline{S_\psi}$ has a nontrivial element $\overline{s}$ and pick a lift $s \in S_\psi$.

Then for each place v of F, we see that identifying $\widehat{\mathbf{U}} \subset {}^L\mathbf{U}$ with $\widehat{\mathbf{U}_v} \subset {}^L\mathbf{U}_v$, we get that $s \in S_{\psi_v}$ so that $(\psi_v, \overline{s}_v) \in \mathcal{SP}_{F_v}(\mathbf{G}_v)$ and hence by I.2.15 we get an endoscopic datum (H_v, s_v, η_v) of $\mathbf{G}_v$. Under our identifications, $\widehat{H_v} \subset {}^L\mathbf{G}_v$ and η_v is the inclusion map. Moreover $\eta_v(s_v) = s$. In particular, we have for all v that $\eta_v(\widehat{H_v}) = Z_{\widehat{\mathbf{G}}}(s)^0$.

By I.6.1, we get a datum $[\mathbf{H}, s, {}^L\eta, \psi^{\mathfrak{e}}] \in \mathcal{AEP}_F(\mathbf{U})$. In particular, we have a global endoscopic datum $(\mathbf{H}, s, \eta)$ that localizes at each place v to (H_v, s_v, η_v). Now, v_∞ ramifies over E since E/F is imaginary and hence $\mathbf{U}_{v_\infty}$ is an inner form of $U_{E_{v_\infty}/F_{v_\infty}}(n)$. Since we assumed ψ_{v_∞} is elliptic, it follows from I.2.15 that $(H_{v_\infty}, s_{v_\infty}, \eta_{v_\infty})$ is an elliptic endoscopic datum.

We now pick a lift ${}^L\eta$ of η and note that for each place v, we get a map ${}^L\eta_v$. Now, we recall that the choice of the lift ${}^L\eta_v$ in the construction of the map $\mathcal{SP}_F(\mathbf{G}_v) \to \mathcal{EP}_F(\mathbf{G}_v)$ is arbitrary and picking a different lift does not change $(\mathbf{H}_v, s, \eta_v)$. In particular, we could have picked at each place v, the lift ${}^L\eta_v$ of η_v that we got from localizing ${}^L\eta$. Note however that doing so does change the parameters $\psi^{\mathbf{H}_v}$.

In particular, we now have, without loss of generality, a tuple $(\mathbf{H}, s, {}^L\eta)$ and for each $v \in F$, a parameter $\psi^{\mathbf{H}_v}$ of $\mathbf{H}_v$ such that ${}^L\eta_v \circ \psi^{\mathbf{H}_v}$ is relevant. Furthermore, since ψ_∞ was assumed to be elliptic, $(\mathbf{H}_\infty, s, \eta_v)$ is elliptic. Furthermore, $\mathbf{H}$ is a product of unitary groups and so has an elliptic maximal torus. In particular, we are now in the situation to apply I.2.17. We get that there exists a semisimple $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \in \mathcal{RE}(\mathbf{U})$. Now by I.2.6 we get an element $(\gamma, \kappa) \in \mathcal{SS}_F^{\mathrm{ell}}(\mathbf{G})$. Since s is nontrivial in $\overline{S_\psi}$, it follows that κ is nontrivial. This contradicts that for $\mathbf{U}$, all $\mathfrak{K}(I_\gamma/F)$ are trivial. □

I.6.2 Isotypic Components

Now, let $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ and choose χ_κ, Ξ for $\mathbf{U}$ as in [KMSW14, Thm. 1.7.1]. Then it follows from that theorem that we have a decomposition

$$L^2_{\mathrm{disc}}(\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}_F)) = \bigoplus_{\psi \in \Psi_2(\mathbf{U}^*, \eta_{\chi_\kappa})} \bigoplus_{\pi \in \Pi_\psi(\mathbf{U}, \omega, \epsilon_\psi)} \pi. \tag{95}$$

Now we fix a representation $\boldsymbol{\pi}$ of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$ that is discrete at ∞. Since $\mathbf{G}(\mathbb{A}_\mathbb{Q}) = \mathbf{U}(\mathbb{A}_F)$, we can equivalently consider $\boldsymbol{\pi}$ to be a representation of $\mathbf{U}(\mathbb{A}_F)$. We call this representation $\boldsymbol{\pi}'$ so as to avoid confusion. Now, at any place p of $\mathbb{Q}$, we have

$$\boldsymbol{\pi}_p = \bigotimes_{v|p} \boldsymbol{\pi}'_v. \tag{96}$$

Then the Satake parameters of $\boldsymbol{\pi}'$ determine a unique parameter $\psi_{\boldsymbol{\pi}'}$ of $\mathbf{U}$ such that $\boldsymbol{\pi}' \in \Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U}, \xi)$. Since $\boldsymbol{\pi}'$ is discrete at each infinite place, it follows that $\psi_{\boldsymbol{\pi}'}$ has trivial Arthur SL_2-factor and hence is generic. Hence by the comment after equation [KMSW14, (1.2.4)], we have that each element of $\Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U},\omega)$ is irreducible. Moreover each element of the packet appears with multiplicity 1 by the global multiplicity formula.

Now by I.6.2, it follows that $\Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U},\omega,\epsilon_{\psi_{\boldsymbol{\pi}'}}) = \Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U},\omega)$ or, in other words, the condition involving $\epsilon_{\psi_{\pi'}}$ is vacuous. In particular, if we let $\boldsymbol{\pi}'^p$ denote the factor of $\boldsymbol{\pi}$ that is the complement of $\bigotimes_{v|p} \boldsymbol{\pi}'_v$, then we have

$$L^2_{\mathrm{disc}}(\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}_F))[\boldsymbol{\pi}'^p] = \bigotimes_{v|p} \bigoplus_{\pi'_v \in \Pi_{\psi_{\boldsymbol{\pi}'_v}}(\mathbf{U}_v,\omega)} \pi'_v \tag{97}$$

We can define a parameter $\psi_{\boldsymbol{\pi}}$ of the group $\mathbf{G}$. Since $\mathbf{G}_p = \prod_{v|p} \mathbf{U}_v$, it follows that

$$\bigoplus_{\pi_p \in \Pi_{\psi_{\boldsymbol{\pi}_p}}(\boldsymbol{G}(\mathbb{Q}_p),\omega)} \pi_p = \bigotimes_{v|p} \bigoplus_{\pi'_v \in \Pi_{\psi_{\boldsymbol{\pi}'_v}}(\mathbf{U}_v,\omega)} \pi'_v. \tag{98}$$

In particular, we record the following result.

Lemma I.6.3. *We have the following decomposition.*

$$L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\boldsymbol{\pi}^p] = \bigoplus_{\psi_{\boldsymbol{\pi}_p} \in \Pi_{\psi_p}(\boldsymbol{G}(\mathbb{Q}_p),\omega)} \pi_p. \tag{99}$$

Part II

The ℓ-adic cohomology of compact Shimura varieties with no endoscopy

II.1 Introduction

We state in this section a result on the decomposition of the cohomology of certain compact Shimura varieties $\mathsf{Sh}(\mathbf{G}, X)$ in the case when $(\mathbf{G}, X)$ has no relevant global endoscopy (in the sense of §I.5). The results here are largely a technical generalization of the results in [Kot92a] using the newly proven results of [KSZ] checking, in all cases, that the methods of [Kot92a] work in this more general setting under the umbrella assumption of no endoscopy.

This decomposition will be key to understanding the Scholze–Shin conjecture at a given bad place in terms of the already established Scholze–Shin conjecture at a good place which, at least in the case of the trivial endoscopic triple, is just a rephrasing of the results of [Kot84a].

II.2 Statement of the decomposition result

Let us now state the decomposition result of interest to us. To do this, we begin by detailing the necessary setup.

We start with a Shimura datum $(\mathbf{G}, X)$ which we assume to be of abelian type. We assume further that our group satisfies Axiom **SV**5 of [Mil04]. By [Mil04, Theorem 5.26] (and the succeeding discussion) this is equivalent to assuming that $(A_{\mathbf{G}})_{\mathbb{R}} = A_{\mathbf{G}_{\mathbb{R}}}$. We assume further that $\mathbf{G}/Z(\mathbf{G})$ is $\mathbb{Q}$-anisotropic. Note that this implies that if $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ then $\mathbf{T}_{\mathbb{R}}$ is an elliptic maximal torus in $\mathbf{G}_{\mathbb{R}}$. Thus, in particular, we see that $\mathbf{G}(\mathbb{R})$ has discrete series (see [Kna01, Theorem 12.20]). We also assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected.

Most importantly, we assume that the group $\mathbf{G}$ has no relevant global endoscopy (in the sense of §I.5). This is the key assumption which makes the proof of Theorem II.2.1 below possible.

Let us fix a prime ℓ and let ξ be an algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbf{G}$ (i.e. an algebraic representation $\xi : \mathbf{G}_{\overline{\mathbb{Q}_\ell}} \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ for some $\overline{\mathbb{Q}_\ell}$-space V) which induces a representation

$$\mathbf{G}(\mathbb{A}_f) \xrightarrow{\text{proj.}} \mathbf{G}(\mathbb{Q}_\ell) \hookrightarrow \mathbf{G}(\overline{\mathbb{Q}_\ell}) \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$$

which we also denote ξ.

Let us also note that from the conjugacy class X one obtains a conjugacy class of cocharacters $\boldsymbol{\mu}$ of $\mathbf{G}_{\mathbb{C}}$ as on [Mil04, Page 111] which (as in loc. cit.) induces a unique conjugacy class of cocharacters, also denoted $\boldsymbol{\mu}$, over $\overline{\mathbb{Q}}$. Moreover, by definition, the reflex field $E(\boldsymbol{G}, X)$ is precisely the reflex field of $\boldsymbol{\mu}$ as in §IV.1.4. We denote this field by $\mathbf{E}_{\boldsymbol{\mu}}$. Then, by the contents of §IV.1.4 we obtain a representation $r_{\boldsymbol{\mu}} : \widehat{G} \rtimes W_{\mathbf{E}_{\boldsymbol{\mu}}} \to \mathrm{GL}(V(\boldsymbol{\mu}))$.

Finally, fix an isomorphism $\iota_l : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ which we implicitly use throughout the sequel. In particular, via ι_ℓ we get an algebraic representation $\xi_{\mathbb{C}}$ over $\mathbb{C}$.

With these assumptions, and in the notation as above the following holds:

Theorem II.2.1. *There is a decomposition of virtual* $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)\times W_{\mathbf{E}_{\boldsymbol{\mu}}}]$*-representations*

$$H^*(\mathsf{Sh},\mathcal{F}_\xi)=\bigoplus_{\pi_f}\pi_f\boxtimes\sigma(\pi_f),\tag{100}$$

where π_f *ranges over admissible* $\overline{\mathbb{Q}_\ell}$*-representations of* $\mathbf{G}(\mathbb{A}_f)$ *such that there exists an automorphic representation* π *of* $\mathbf{G}(\mathbb{A})$ *such that;*

1. $\pi_f\cong(\pi)_f$ *(using our identification* $\overline{\mathbb{Q}}_\ell\cong\mathbb{C}$*)*
2. $\pi_\infty\in\Pi_\infty(\xi)$.

Moreover, for each π_f *there exists a cofinite set* $S(\pi_f)\subseteq S^{\mathrm{ur}}(\pi_f)$ *of primes* p *such that for each prime* $\mathfrak{p}$ *over* $\mathbf{E}_{\boldsymbol{\mu}}$ *lying over* p *and each* $\tau\in W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ *the following equality holds:*

$$\mathrm{tr}(\tau\mid\sigma(\pi_f))=a(\pi_f)\,\mathrm{tr}(\tau\mid r_{-\boldsymbol{\mu}}\circ\varphi_{\pi_p})p^{\frac{1}{2}v(\tau)[\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}:\mathbb{Q}_p]\dim\mathsf{Sh}},\tag{101}$$

for some integer $a(\pi_f)$ *(see Definition II.3.5).*

As stated in the introduction, the proof of this result (closely imitating [Kot92a]) is broken up in to three main steps. These, very roughly, go as follows:

- **Step 1**: Construct a function f which projects the cohomology $H^*(\mathsf{Sh},\mathcal{F}_\xi)$ on to its π_f-isotypic component so that, by construction, the quantity $\mathrm{tr}(f\times\tau\mid H^*(\mathsf{Sh},\mathcal{F}_\xi))$ agrees with left-hand side of (101).
- **Step 2**: Use results of Kisin-Shin-Zhu in [KSZ] to express the quantity $\mathrm{tr}(f\times\tau\mid H^*(\mathsf{Sh},\mathcal{F}_\xi))$ in terms of sums of orbital integrals.
- **Step 3**: Pseudo-stabilize the result to obtain the right-hand side of (101).

The rest of Part I will be dedicated to carefully carrying out this proof step-by-step.

II.3 The function f

In this section we construct a smooth function $f:G(\mathbb{A})\to\mathbb{C}$ alluded to in Step 1 from the previous section. This function f, which will admit a factorization $f=f_\infty f^\infty$, is deceptively notated since it really depends on the following data:

- An admissible $\overline{\mathbb{Q}_\ell}$-representation π_f of $\mathbf{G}(\mathbb{A}_f)$.
- A compact open subgroup K of $G(\mathbb{A}_f)$ such that π_f has a non-zero K-invariant vector.
- The set $\Pi^0_\infty(\xi)$.

The function f will be constructed in a highly non-explicit way. This is relevant since the entrance of the cofinite set $S(\pi_f) \subseteq S^{\mathrm{ur}}(\pi_f)$ in Theorem II.2.1 enters in to the picture via f. Namely hidden in Step 2 of the outline from the previous section is the assumption that at p one can decompose f as $f = f^p \mathbb{1}_{K_{0,p}}$. Thus, the inexplicitness of f is part and parcel with the inexplicitness of the cofinite set $S(\pi_f)$.

II.3.1 The construction of f_∞ and basic properties

Let us begin by recalling the basic setup of the theory of pseudo-coefficients in the context that we need them. Let us fix χ to be a smooth character $A_{\mathbf{G}}(\mathbb{R})^0 \to \mathbb{C}^\times$. We then define the following set:

Definition II.3.1. *The set $\mathscr{H}(\mathbf{G}(\mathbb{R}), \chi)$ is the set of all smooth functions $f : \mathbf{G}(\mathbb{R}) \to \mathbb{C}$ such that*

1. *$f(ag) = \chi(a) f(g)$ for all $a \in A_{\mathbf{G}}(\mathbb{R})^0$.*
2. *The function $f\chi^{-1} : \mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0 \to \mathbb{C}$ is compactly supported.*

Let us now consider the set $\Pi_\infty(\chi)$ of irreducible admissible representations of $\mathbf{G}(\mathbb{R})$ with central character χ and let $\Pi^0_\infty(\chi)$ denote the subset of $\Pi_\infty(\chi)$ consisting of those elements which are discrete series for $\mathbf{G}(\mathbb{R})$. Let us note that for a fixed $\pi^0_\infty \in \Pi^0_\infty(\chi)$ we make the following definition:

Definition II.3.2. *A* pseudo-coefficient *for π^0_∞ is an element $f_{\pi^0_\infty} \in \mathscr{H}(\mathbf{G}(\mathbb{R}), \chi^{-1})$ such that for all tempered $\pi_\infty \in \Pi_\infty(\chi)$ we have that*

$$\operatorname{tr}(f_{\pi^0_\infty} \mid \pi_\infty) = \begin{cases} 1 & \text{if} \quad \pi_\infty \cong \pi^0_\infty \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{102}$$

Let us be clear about what the above trace means. Namely, for π_∞ in $\Pi_\infty(\chi)$ we set $\operatorname{tr}(f_{\pi^0_\infty} \mid \pi_\infty)$ to be the trace of the operator

$$v \mapsto \int_{\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0} f_{\pi^0_\infty}(g) \pi_\infty(g)(v)\, dg \tag{103}$$

which is well-defined since the product of $f_{\pi^0_\infty}$ and π_∞ transform by the identity under $A_{\mathbf{G}}(\mathbb{R})^0$ and since $f_{\pi^0_\infty}$ is compactly supported on $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0$.

The existence of such pseudo-coefficients can be deduced from the research announcement [CD85], with a full proof found in the references of said paper.

Let us now fix an element $\pi_\infty^0 \in \Pi_\infty^0(\xi)$ which, in particular, is an element of $\Pi_\infty^0(\chi_\xi^{-1})$. Let us denote by $f_{\pi_\infty^0} \in \mathscr{H}(\mathbf{G}(\mathbb{R}), \chi_\infty)$ the pseudo-coefficient of π_∞^0 in the sense discussed above.

We record the following equality:

Proposition II.3.3. *For any $\gamma_\infty \in \mathbf{G}(\mathbb{R})$ semisimple, the following equality holds:*

$$\mathrm{SO}_{\gamma_\infty}(g) = \begin{cases} \mathrm{tr}(\xi(\gamma_\infty))\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0/I_\infty(\mathbb{R}))^{-1}e(I_\infty) & \text{if} \quad \gamma_\infty \in \mathbf{G}(\mathbb{R})^{\mathrm{ell}} \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{104}$$

where $g := (-1)^{\dim \mathsf{Sh}} f_{\pi_\infty^0}$ and I_∞ is the unique anisotropic modulo center inner form of I_{γ_∞}.

Remark II.3.4. Note that the existence of I_∞ follows from Lemma IV.1.11. Indeed, since we are assuming that $\mathbf{G}(\mathbb{R})$ has an elliptic maximal torus we know from Corollary IV.1.9 that for $\gamma_\infty \in \mathbf{G}(\mathbb{R})^{\mathrm{ell}}$ we have that $\gamma_\infty \in T(\mathbb{R})$ for some maximal elliptic torus T of $\mathbf{G}_{\mathbb{R}}$. Note then that $T \subseteq I_{\gamma_\infty}$ and thus I_{γ_∞} has an elliptic maximal torus, which shows that Lemma IV.1.11 applies.

Let us note that in the above formula the quantity $\mathrm{SO}_{\gamma_\infty}(g)$ is sensical (in the sense that the integrals defining this stable orbital integral converge) since $f_{\pi_\infty^0}$ is compactly supported on $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^+$ and so, in particular, compactly supported on $I_{\gamma_\infty}(\mathbb{R})\backslash\mathbf{G}(\mathbb{R})$ since $A_{\mathbf{G}}(\mathbb{R})^+ \subseteq I_{\gamma_\infty}(\mathbb{R})$.

Proposition II.3.3. We follow [Kot92a, §3.1]. Let us first assume that γ_∞ is strongly regular. Note that then that since γ_∞ is elliptic strongly regular, we have that $I_{\gamma_\infty} = I_\infty$. Now we have:

$$O_\gamma(f_{\pi_\infty^0}) = \begin{cases} \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\Theta_{\pi_\infty^0}(\gamma_\infty^{-1}) & \text{if} \quad \gamma_\infty \text{ elliptic} \\ 0 & \text{if} \quad \gamma_\infty \text{ not elliptic} \end{cases} \tag{105}$$

where $\theta_{\pi_\infty^0}$ is the function associated to the Harisha-Chandra character of π_∞^0 by Harish-Chandra's theorem (for a proof of this formula see [Art93, Theorem 5.1]). Suppose now that γ_∞ is strongly regular elliptic. Then, by Proposition IV.1.21 we deduce that

$$\begin{aligned} \mathrm{SO}_{\gamma_\infty}(g) &:= \sum_{[\gamma'_\infty]\sim_s[\gamma_\infty]} O_{\gamma'_\infty}(g) \\ &= \sum_{w \in W_{\mathbb{C}}/W_{\mathbb{R}}} O_{w\cdot\gamma_\infty}(g) \\ &= \sum_{w \in W_{\mathbb{C}}/W_{\mathbb{R}}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\Theta_{\pi_\infty^0}(w\cdot\gamma_\infty^{-1})) \end{aligned} \tag{106}$$

Note that in the first line the lack of the terms $a(\gamma'_\infty)$ is due to our assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected, and the lack of the Kottwitz sign is because $I_{\gamma'_\infty}$, by assumption, is a torus which has trivial Kottwitz sign (since it is quasi-split).

Let us write $\pi_\infty^0 := \pi(\varphi, B_0)$ as in [Kot90, Page 185]. Then, this last term is equal, by the Harish-Chandra character formula, to

$$\sum_{w\in W_\mathbb{C}/W_\mathbb{R}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(A_\mathbf{G}(\mathbb{R})^+\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1} \sum_{w'\in W_\mathbb{R}} \chi_{w'\cdot B_0}(w\cdot\gamma_\infty^{-1})\Delta_{w'\cdot B_0}(w\cdot\gamma_\infty^{-1}) \tag{107}$$

But, this is visibly equal to

$$\sum_{w\in W_\mathbb{C}/W_\mathbb{R}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(A_\mathbf{G}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1} \sum_{w'\in W_\mathbb{R}} \chi_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})\Delta_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1}) \tag{108}$$

which is equal to

$$\sum_{w'\in W_\mathbb{R}}\sum_{w\in W_\mathbb{C}/W_\mathbb{R}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(A_\mathbf{G}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1} \chi_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})\Delta_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1}) \tag{109}$$

which, by concatenation, is equal to

$$(-1)^{\dim \mathsf{Sh}} \sum_{w''\in W_\mathbb{C}} \mathrm{vol}(A_\mathbf{G}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1} \chi_{w''\cdot B}(\gamma_\infty^{-1})\Delta_{w''\cdot B_0}(\gamma_\infty^{-1}) \tag{110}$$

But, by the Weyl character formula this is equal to

$$\mathrm{vol}(A_\mathbf{G}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\,\mathrm{tr}\,\xi(\gamma_\infty) \tag{111}$$

as desired.

For the case for general elliptic $\gamma_\infty \in \mathbf{G}(\mathbb{R})$ (not necessarily strongly regular) we proceed as follows. Note that by Corollary IV.1.10 γ_∞ is contained in some elliptic maximal torus of $\mathbf{G}_\mathbb{R}$. The result then follows from the above description and [She83, Lemma 2.9.3].

Note that the Kottwitz sign $e(I_\infty)$ enters due to the difference in sign conventions between this article and that of Shelstad (see [She83, Page 2.12]). $\square$

With the above, we are now well-positioned to define f_∞ and observe its basic properties. Namely, let us define f_∞ as follows:

$$f_\infty := \frac{(-1)^{\dim \mathsf{Sh}}}{|\Pi_\infty^0(\xi)|}\sum_{\pi_\infty^0\in\Pi_0^\infty} f_{\pi_\infty^0} \tag{112}$$

Note that this sum is sensical since $\Pi_\infty^0(\xi)$ is a finite set.

Note then that by the definition of pseudo-coefficients we have that for any π_∞ an irreducible tempered representation of $\mathbf{G}(\mathbb{R})$ in $\Pi_\infty(\xi)$, the following equality holds:

$$\mathrm{tr}(f_\infty \mid \pi_\infty) = \begin{cases} \dfrac{(-1)^{\dim \mathsf{Sh}}}{|\Pi_\infty^0(\xi)|} & \text{if} \quad \pi_\infty \in \Pi_\infty^0(\xi) \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{113}$$

with this trace having the same meaning as the discussion succeeding Definition II.3.2.

The last thing we record is that there is a certain well-defined integer $a(\pi_f)$ associated to any admissible irreducible representation π_f of $\mathbf{G}(\mathbb{A}_f)$. Namely, let us make the following definition.

Definition II.3.5. *Let notation be as above. We then define $a(\pi_f)$ as follows:*

$$a(\pi_f) := \sum_{\pi_\infty \in \Pi_\infty(\xi)} m(\pi_f \otimes \pi_\infty)\,\mathrm{tr}(f_\infty \mid \pi_\infty) \tag{114}$$

Let us begin by observing the following:

Lemma II.3.6. *The equality*

$$a(\pi_f) = \sum_{\pi_\infty \in \Pi_\infty(\xi)} m(\pi_f \otimes \pi_\infty)\,\mathrm{tr}((-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty} \mid \pi_\infty) \tag{115}$$

holds for any $\pi^0_\infty \in \Pi^0_\infty(\xi)$.

Proof. Let K be any compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$. Let us then note that the $\mathbb{C}$-space V of automorphic representations such that $\varpi_\infty \in \Pi_\infty(\xi)$ is an admissible $\mathbf{G}(\mathbb{A}_f)$-representation by Harish-Chandra's theorem (e.g. see [BJ79, Theorem 1.7]). Let us choose any function h as in Proposition II.3.11 where normalize so that $\mathrm{tr}(h \mid \pi_f) = 1$. Note then that our desired equality is equivalent to

$$\sum_\pi m(\pi)\,\mathrm{tr}(hf_\infty \mid \pi) = \sum_\pi m(\pi)\,\mathrm{tr}(h(-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty} \mid \pi) \tag{116}$$

where π travels over automorphic $\mathbf{G}(\mathbb{A}_f)$-representations with central character agreeing with that of $\xi^\vee_\mathbb{C}$. But, by Proposition IV.2.16 this is equivalent to the claim that

$$\tau(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\text{s.s.}}} SO_\gamma(hf_\infty) = \tau(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\text{s.s.}}} SO_\gamma(h(-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty}) \tag{117}$$

But, note that the left-hand side of this equality is equal, by definition of f_∞, to

$$|\Pi^0_\infty(\xi)|\tau(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\text{s.s.}}} \sum_{\pi_\infty} SO_\gamma(h(-1)^{\dim \mathsf{Sh}} f_{\pi_\infty}) \tag{118}$$

Note though that by Proposition IV.2.16 we have that

$$SO_\gamma(h(-1)^{\dim \mathsf{Sh}} f_{\pi_\infty}) = SO_\gamma(h(-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty}) \tag{119}$$

(because both sides are equal to the expression given in Proposition IV.2.16) from where the conclusion follows. □

The following proposition will be useful shortly:

Proposition II.3.7. *The complex number $a(\pi_f)$ is an element of $\mathbb{Z}$.*

Proof. It suffices to show that if $f_{\pi_\infty^0}$ is a pseudo-coefficient for an element $\pi_\infty^0 \in \Pi_\infty^0(\xi)$ then $\mathrm{tr}(f_{\pi_\infty^0} \mid \pi_\infty) \in \mathbb{Z}$ for every $\pi_\infty \in \Pi_\infty(\xi)$. Suppose that π_∞ has the same central character as π_∞^0. We know that π_∞, as an element of the Grothendieck group of representations of $\mathbf{G}(\mathbb{R})$, is a $\mathbb{Z}$-linear combination of standard representations (e.g. see [ABV12, Lemma 1.20]. We then use the fact (see [CD90, Corollaire Page 213]) that the trace of a pseudocoefficient for π_∞^0 is 0 on all standard representations except π_∞^0. $\square$

Finally, we record the following alternative description of the integer $a(\pi_f)$:

Proposition II.3.8. *We have an equality*

$$a(\pi_f) = \sum_{\pi_\infty \in \Pi_\infty^0} m(\pi_f \otimes \pi_\infty) N^{-1} \mathrm{ep}(\pi_\infty \otimes \xi_\mathbb{C}) \tag{120}$$

where $N = |\Pi_\infty^0| \cdot |\pi_0(\mathbf{G}(\mathbb{R})/Z(\mathbf{G})(\mathbb{R}))|$ and $\mathrm{ep}(\pi_\infty \otimes \xi_\mathbb{C})$ is the Euler-Poincare characteristic of $H^*(\mathfrak{g}, K_\infty, \pi_\infty \otimes \xi_\mathbb{C})$.

Proof. See [Kot92a, Lemma 3.2] and [Kot92a, Lemma 4.2]. The only assertion thatr used in the proof that requires justifcation is the fact that $K_\infty/Z(\mathbf{G})(\mathbb{R})$ is connected in our situation. But, this follows from the observation that if K'_∞ is a maximal compact subgroup of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ (which is connected by [PS92, Theorem 7.6] given our assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected) then K'_∞ surjects on to $K_\infty/Z(\mathbf{G})(\mathbb{R})$. $\square$

Corollary II.3.9. *Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$. Then, $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)[\pi_f^K] \neq 0$ if and only if $a(\pi_f) \neq 0$.*

Proof. This follows from [BR94, frm-e.3] as well as [BC+83]. Again, note that by our assumption that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic, we know that $\mathsf{Sh}_M(\mathbf{G}, X)^{\mathrm{an}}_\mathbb{C}$ is proper for all neat $M \subseteq \mathbf{G}(\mathbb{A}_f)$ (by [Pau04, Lemma 3.1.5]) and so L^2-cohomology agrees with singular cohomology, and thus has a comparison with étale cohomology by Artin's comparison theorem. $\square$

Finally, we record the following result of Vogan-Zuckerman. Namely:

Proposition II.3.10 ([VZ84])**.** *Suppose that ξ is regular. Then, we have the equality $a(\pi_f) = (-1)^{\dim \mathsf{Sh}} m(\pi_f \otimes \pi_\infty^0)$.*

II.3.2 The construction of f^∞

To construct f^∞ we first start with the following basic observation:

Proposition II.3.11. *Let $K \subseteq \mathbf{G}(\mathbb{A}_f)$ be compact open and let V be an admissible semisimple $\overline{\mathbb{Q}_\ell}[G(\mathbb{A}_f)]$-representation. Then, there exists some $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ such that the action of P on V is the projector of V onto $V^K[(\pi^\infty)^K]$.*

Proof. This follows immediately from the general version of the Jacobson Density Theorem (e.g. as in [Lor07, F20]). Namely, if we decompose

$$V^K = \bigoplus_i V_i^{e_i} \tag{121}$$

where V_i are the simple components of $\mathscr{H}(G(\mathbb{A}_f), K)$ then by loc. cit. we can find some element $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ such that the image of P in $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ is the projector of V^K onto $V^K[(\pi^\infty)^K]$. Noting then that since $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ we have that $P = Pe_K$ and noting that e_K projects V onto V^K, the conclusion follows. $\square$

We can then construct the function f^∞ by taking P to be any element of $\mathscr{H}(\mathbf{G}(\mathbb{A}_f), K)$ from the previous proposition where we take

$$V := \bigoplus_{i=1}^{2\dim(\mathsf{Sh})} H^i(\mathsf{Sh}, \mathcal{F}_\xi) \tag{122}$$

To do this, it suffices to show that V is semisimple and admissible. For the first property note that since $\mathsf{Sh} \to \mathsf{Sh}_K$ is a pro-finite Galois cover, the Leray spectral sequence implies that

$$V^K = H^i(\mathsf{Sh}, \mathcal{F}_\xi)^K = H^i(\mathsf{Sh}_K, \mathcal{F}_\xi) \tag{123}$$

the latter term of which is finite-dimensional by standard algebraic geometry. For the second property we use the following well-known result:

Theorem II.3.12. *For all $i \geqslant 0$ The admissible $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-representation $H^i(\mathsf{Sh}, \mathcal{F}_\xi)$ is semisimple.*

Proof. It suffices to show, by Artin's comparison theorem, that for any embedding of E into $\mathbb{C}$ that the $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-representation

$$\varinjlim_{K} H^i_{\mathrm{sing}}(\mathsf{Sh}^{\mathrm{an}}_{K,\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_{\xi,K}) \tag{124}$$

is semisimple. This follows at once from [BR94, §2.3] as well as [BC+83]. Note that since $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic that $\mathsf{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ is compact for all K (see [Pau04, Lemma 3.1.5]), and thus the L^2-cohomology of $\mathsf{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ agrees with singular cohomology. $\square$

Let us note that for any f^∞ defined as above we can renormalize such that for π'_f any admissible $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-representation for which the space $H(\mathsf{Sh}_K, \mathcal{F}_\xi)[(\pi'_f)^K]$ is non-zero then

$$\operatorname{tr}(f^\infty \mid \pi'_f) = \begin{cases} 1 & \text{if} \quad \pi_f \cong \pi'_f \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{125}$$

In the sequel we fix such a function f^∞. It is worth noting that we cannot specify the trace of f^∞ on representations whose K-invariants do not appear in $H(\mathsf{Sh}_K, \mathcal{F}_\xi)$. It is also noting that f^∞ is not unique. This non-unicity will be a non-issue for us, and so we have chosen to not notate the non-unicity of f.

II.4 A geometric trace formula in the case of good reduction

We recall here the statement of the relevant version of the main formula from [KSZ] necessary to prove Theorem II.2.1. We keep the assumptions from §II.2 although the only pivotal assumption for the version of the results of [KSZ] that we use is the assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected.

Let us fix the notation as in §II.2. We also fix the following extra notation. Let us fix a prime $p \in S(\mathbf{G})$. Fix a finite place $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p. Since $\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}/\mathbb{Q}_p$ is unramified (by Corollary IV.1.30) we know that $\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}} \cong \mathbb{Q}_{p^r}$ for some $r \geqslant 1$. Fix $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ to be a neat compact open subgroup and set $K := K^p K_{0,p}$.

Before we proceed let us make the following observation:

Lemma II.4.1. *For $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ sufficiently small the group $Z(\mathbb{Q})_K$ is trivial.*

Proof. Let us note that since we are assuming that $(A_\mathbf{G})_\mathbb{R} = A_{\mathbf{G}_\mathbb{R}}$ that for all sufficiently small compact open subgroups K_1 of $G(\mathbb{A}_f)$ we have that $Z(\mathbb{Q})_{K_1}$ is trivial (e.g. see [Mil04, Remark 5.27]). Note then that by possibly shrinking K_1, we may assume that $K_1 = K_1^p K_p$ with $K_p \subseteq K_{0,p}$. Since $K_p \subseteq K_{0,p}$ is of finite index, $Z(\mathbb{Q})_{K_1^p K_{0,p}}$ is finite. Now, since $Z(\mathbb{Q})$ embeds diagonally into $\mathbf{G}(\mathbb{A}_f)$, we can shrink K_1^p to some K^p such that $Z(\mathbb{Q})_{K^p K_{0,p}}$ is trivial as desired. ☐

Given this lemma we assume, in all future discussion, that K is small enough so that $Z(\mathbb{Q})_K$ is the trivial group.

We continue, as in [KSZ, §5.5], to fix the following extra data/notation:

- Fix $j \geqslant 1$ and set $n := rj$.

- Let $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ be a (equivalence class of) degree n (punctual) Kottwitz triple(s) as in [KSZ, Definition 2.7.1] or [Kot90, Page 165].

- For such a Kottwitz triple $\mathfrak{t} = (\gamma_0, (\gamma_\ell), \delta)$ set $I_0(\mathfrak{t}) := I_{\gamma_0}$ and for each place v of $\mathbb{Q}$ set $I_v(\mathfrak{t})$ to be the inner form of $(I_0(\mathfrak{t}))_v$ as in [KSZ, §4.7.18] (see also [Kot90, Page 169] and [Kot90, Page 171]).

- Let us denote by $I(\mathfrak{t})$ the unique inner form of I_0 such that $I(\mathfrak{t})_v \cong I_v(\mathfrak{t})$ for all v (e.g. see [KSZ, Proposition 4.7.19] and [Kot90, Page 171]).

- Let $\alpha(\mathfrak{t}) \in \mathfrak{K}(I_0/\mathbb{Q})^D$ as in [Kot90, §2] and [KSZ, 4.7.13].

- Set $R := \mathsf{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} \mathbf{G}_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ and let θ be the automorphism of R corresponding to the Frobenius element of $\mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$. Let $R_{\delta \rtimes \theta}$ be as [KSZ, Definition 1.5.1]. Namely, for a $\mathbb{Q}_p$-algebra A we set

$$R_{\delta \rtimes \theta}(A) = \{g \in \mathbf{G}(A \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^n}) : g\delta\sigma(g)^{-1} = \delta\} \qquad (126)$$

- Let us a fix a Haar measures dg^p on $\mathbf{G}(\mathbb{A}_f^p)$ arbitrarily and a Haar measure dg_p on $R(\mathbb{Q}_p)$ where we require that the mass of $R(\mathbb{Z}_p)$ is 1.

- Also choose Haar measures on $I_p = I(\mathbb{Q}_p)$ and $I(\mathbb{A}_f^p)$. Note that we have an isomorphism $I_p \cong R_{\delta \rtimes \theta}$ and for all $\ell \neq p$ we also have isomorphisms $I_\ell \cong Z_G(\gamma_\ell)$. Having fixed such isomorphism we can transfer these Haar measures to Haar measures on $R_{\delta \rtimes \theta}(\mathbb{Q}_p)$ and $I_\gamma(\mathbb{A}_f^p)$.

- Let $\mu : \mathbb{G}_{m,\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}} \to \mathbf{G}_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ be any element of $\boldsymbol{\mu}_\mathfrak{p}$.

- Let us denote by ϕ_n denote $\mathbb{1}_{R(\mathbb{Z}_p)\mu(p)^{-1}R(\mathbb{Z}_p)}$.

- We define the twisted orbital integral

$$TO_\delta(\phi_n) := \int_{R_{\delta \rtimes \theta}(\mathbb{Q}_p) \backslash R(\mathbb{Q}_p)} \phi_n(g^{-1}\delta\sigma(g))dg. \qquad (127)$$

- Define $c_1(\mathfrak{t}) := \mathrm{vol}(I(\mathbb{Q})Z_K \backslash I(\mathbb{A}_f))$.

- Set $c_2(\mathfrak{t}) = |\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, \mathbf{G}))|$.

- Set $c(\mathfrak{t}) := c_1(\mathfrak{t})c_2(\mathfrak{t})$.

We then state the main result of [KSZ] specialized to our current situation:

Theorem II.4.2 ([KSZ, Theorem 5.5.2])**.** *For sufficiently small K^p, we have the following. Let $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p), K^p)$. Normalize the action of $f^p dg^p$ on $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)$ such that $\mathrm{vol}_{dg^p}(K^p)^{-1} 1_{K^p} dg^p = 1$. Then the quantity*

$$\mathrm{tr}(\Phi^j \times \mathbb{1}_{K_{0,p}} f^p dg^p \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) \qquad (128)$$

is equal to

$$\sum_{\substack{\mathfrak{t}=(\gamma_0,\gamma,\delta)\\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\mathfrak{t})O_\gamma(f^p)TO_\delta(\phi_n)\operatorname{tr}\xi(\gamma_0) \tag{129}$$

Proof. In the following we merely justify the simplifications to [KSZ, Theorem 5.5.2] made in the above.

First let us note that since $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic that Sh_K is proper (e.g. [Pau04, Lemma 3.1.5] noting that $\mathbf{G}$ being $\mathbb{Q}$-anisotropic is equivalent to $G(\mathbb{Q})$ containing no unipotent elements by [BT72, §8]). This obviously allows us to replace compactly supported cohomology by normal étale cohomology.

This observation also allows us to take $j = 1$ (or $m = 1$ in the notation of [KSZ]). Indeed, the proof of [KSZ, Theorem 5.5.2] uses the Fujiwara-Varshavsky trace formula which requires that j is sufficiently large. But, in [Var07, Theorem 2.3.2, c)] a bound is given on permissible j that, in particular, implies that j need only be at least 1 if the integral canonical model Sh_K is proper and the Hecke correspondence is étale. The latter is clear (e.g. see [Kis10, Theorem 2.3.8]). The former follows in the Hodge type case by work of Madapusi-Pera (e.g. see [Per12, Corollary 4.1.7]) and follows in the abelian type case by reduction to geometric connected components and using the fact that such components admit finite surjections from components of Hodge type Shimura varieties.

Next note that having shrunk K^p sufficiently small we have, by Lemma II.4.1, that $Z(\mathbb{Q})_K$ is trivial. This allows us to ignore the stipulations about ξ present in [KSZ, §5.5] as well as replace the set $(\mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})^{\mathrm{ell}})\backslash Z(\mathbb{Q})_K$ (i.e. $\Sigma_{Z(\mathbb{Q})_K,\mathbb{R}\text{-ell}}(\mathbf{G})$ in the notation of [KSZ, §5.5]) with $\mathbf{G}(\mathbb{Q})\cap\mathbf{G}(\mathbb{R})^{\mathrm{ell}}$ (i.e. $\Sigma_{\mathbb{R}\text{-ell}}$ in the notation of [KSZ, §5.5]). This is what allows us to combine the double sum in [KSZ, Theorem 5.5.2] into a single sum of Kottwitz triples.

The absence of the terms $\iota_{\mathbf{G}}(\gamma_0)$ and $\bar{\iota}_{\mathbf{G}}(\gamma_0)$ is explained by the assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected. This assumption also explains the lack of connected components on our R-groups. Indeed, note that $R_{\delta\rtimes\theta}$ is connected since it's an inner form of $Z_{\mathbf{G}}(\gamma)_{\mathbb{Q}_p}$ by [KSZ, Lemma 1.5.3].

The last thing to note is the usage of degree n classical (or punctual in the language of [KSZ]) Kottwitz triples instead of p^n-admissible cohomological Kottwitz triples as is written in [KSZ, Theorem 5.5.2]. The reason that this is permissible is that the natural map from such p^n-cohomological Kottwitz triples to degree n classical Kottwitz triples is a bijection (e.g. see [KSZ, 4.7.12]) and the fact that the term $O(\gamma_0, \alpha^p, [b])$ (as in loc. cit.) associated to a p^n-admissible cohomological Kottwitz triple $(\gamma_0, \alpha^p, [b])$ is defined in terms of the associated degree n Kottwitz triple. A similar statement holds for the Kottwitz invariant $\alpha(\mathfrak{t})$. ∎

II.5 Proof of Theorem II.2.1

We are now prepared to combine the material from the last two subsections, together with the contents of IV.2, to prove our desired claim.

We first prove the following, analogizing the results in [Kot92a, §5]:

Theorem II.5.1. *For all $j \geqslant 1$ and all $f = f^p \mathbb{1}_{K_{0,p}} f_\infty$ where f^p is an element of $\mathscr{H}(G(\mathbb{A}_f^p), K^p)$ and f_∞ is as in §II.3 the following equality holds*

$$\operatorname{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}}) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \tau_K(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\text{s.s.}}} \mathrm{SO}_\gamma(f^p f_n f_\infty) \quad (130)$$

Here we denote by $\tau_K(\mathbf{G})$ the number

$$\tau_K(\mathbf{G}) := \operatorname{vol}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / Z_K A_{\mathbf{G}}(\mathbb{R})^0) \quad (131)$$

which is sensical since $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / Z_K A_{\mathbf{G}}(\mathbb{R})^0$ has finite volume as it is a quotient of $[G]$. Also, f_n denotes the unramified base change of ϕ_n along $G_{\mathbb{Q}_p} \to \mathsf{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} G_{\mathbb{Q}_{p^n}}$ (see the proof of Theorem II.5.1 for details of the definition).

Before we begin, it's useful to note the following lemma:

Lemma II.5.2. *For any classical degree n Kottwitz triple $\mathfrak{t} = (\gamma_0, \gamma, \delta)$ we have that*

$$c(\mathfrak{t}) = \tau_K(\mathbf{G}) \operatorname{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} \quad (132)$$

Here I_∞ is as in Lemma II.3.3.

Proof. This is [KSZ, 6.1.1]. □

Proof. (Proof of Theorem II.5.1) Let us begin by noting that by Theorem II.4.2 in conjunction with Lemma II.5.2

$$\operatorname{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}}) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) \quad (133)$$

is equal to

$$\tau_K(\mathbf{G}) \sum_{\substack{\mathfrak{t} = (\gamma_0, \gamma, \delta) \\ \alpha(\gamma_0, \gamma, \delta) = 1}} \operatorname{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} O_\gamma(f^p) TO_\delta(\phi_n) \operatorname{tr} \xi(\gamma_0) \quad (134)$$

Note though that since $\alpha(\gamma_0, \gamma, \delta)$ is a character of $\mathfrak{K}(\gamma_0, \mathbf{G}, F)$, which is trivial by our assumption that $\mathbf{G}$ has no relevant global endoscopy, we can rewrite this as

$$\tau_K(\mathbf{G}) \sum_{\mathfrak{t} = (\gamma_0, \gamma, \delta)} \operatorname{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} O_\gamma(f^p) TO_\delta(\phi_n) \operatorname{tr} \xi(\gamma_0) \quad (135)$$

Also, note that since $\alpha(\mathfrak{t}) = \alpha(\gamma_0, \gamma, \delta) = 1$ we know by [KSZ, Proposition 4.7.19] that there exists some reductive group $I(\mathfrak{t})$ over $\mathbb{Q}$ such that we have isomorphisms $I(\mathfrak{t})_v \cong I_v(\mathfrak{t})$ for all $v \neq p, \infty$, $I(\mathfrak{t})_p \cong R_{\delta \rtimes \theta}$, and $I(\mathfrak{t})_\infty \cong I_\infty$ where I_∞ is the inner form of $(I_{\gamma_0})_{\mathbb{R}}$ from Proposition IV.2.16. So then we know that

$$e(I_\delta) \prod_{v \neq p, \infty} e(\gamma_v) e(I_\infty) = e(I) = 1 \tag{136}$$

Thus, we may rewrite this sum as

$$\tau_K(\mathbf{G}) \sum_{\mathfrak{t}=(\gamma_0,\gamma,\delta)} \operatorname{vol}(A_{\mathbf{G}}(\mathbb{R})^0 / I_\infty(\mathbb{R}))^{-1} \prod_{v \neq p,\infty} e(\gamma_v) O_\gamma(f^p) e(I_\delta) TO_\delta(\phi_n) e(I_\infty) \operatorname{tr} \xi(\gamma_0) \tag{137}$$

Now, by Proposition II.3.3 we know that

$$\operatorname{tr}(\xi(\gamma_0)) = \operatorname{vol}(A_{\mathbf{G}}(\mathbb{R})^+ / I_\infty(\mathbb{R})) e(I_\infty) SO_{\gamma_0}(f_\infty) \tag{138}$$

So that our sum becomes(noting that the two copies of $e(I_\infty)$ cancel):

$$\tau_K(\mathbf{G}) \sum_{\mathfrak{t}=(\gamma_0,\gamma,\delta)} \prod_{v \neq p,\infty} e(\gamma_v) O_\gamma(f^p) e(I_\delta) TO_\delta(\phi_n) SO_{\gamma_0}(f_\infty) \tag{139}$$

Let us denote by b the base change morphism

$$\mathscr{H}(\mathbf{G}(\mathbb{Q}_{p^n}), \mathcal{G}_p(\mathbb{Z}_{p^n})) \to \mathscr{H}(\mathbf{G}(\mathbb{Q}_p), K_{0,p}) \tag{140}$$

as in the introduction [Kot86a]. One then knows that, by [Lab90, prop 3] (see also [Clo90, thm 1.1]), that

$$\sum_{\substack{\delta \in G(\mathbb{Q}_{p^n})/\sim_\sigma \\ N(\delta) \sim \gamma_0}} e(\delta) TO_\delta(\phi_n) = SO_{\gamma_0}(f_n) \tag{141}$$

Thus, we see that we can rewrite our sum as

$$\tau_K(\mathbf{G}) \sum_{(\gamma_0,\gamma)} \prod_{v \neq p,\infty} e(\gamma_v) \mathrm{O}_\gamma(f^p) SO_{\gamma_0}(f_n) SO_{\gamma_0}(f_\infty) \tag{142}$$

But, by the definition of a stable orbital integral on $\mathbb{A}_f^p$, we see that we can rewrite this as

$$\tau_K(\mathbf{G}) \sum_{\gamma_0} SO_{\gamma_0}(f^p) SO_{\gamma_0}(f_n) SO_{\gamma_0}(f_\infty) = \tau_K(\mathbf{G}) \sum_{\gamma_0} SO_{\gamma_0}(f^p f_n f_\infty) \tag{143}$$

Now, note that while γ_0 a priori only runs over the elements of $\mathbf{G}(\mathbb{Q})$ which are elliptic in $\mathbf{G}(\mathbb{R})$, note that by Proposition II.3.3 we have that $SO_{\gamma_0}(f_\infty)$ is zero for γ_0 not elliptic in $\mathbf{G}(\mathbb{R})$. Thus, we can actually equate this sum to

$$\tau_K(\mathbf{G}) \sum_{\gamma_0 \in \{\mathbf{G}\}_s^{\mathrm{s.s.}}} SO_{\gamma_0}(f^p f_n f_\infty) \tag{144}$$

from where the conclusion follows. □

We are now in a position to apply Proposition IV.2.16 to the above to obtain (keeping the notation of Theorem II.5.1)

$$\begin{aligned}\mathrm{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}})) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) &= \tau_K(\mathbf{G})/\tau(\mathbf{G}) \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi)\,\mathrm{tr}(f \mid \pi) \\ &= \mathrm{vol}(Z_K/Z(\mathbb{Q})_K)^{-1} \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi)\,\mathrm{tr}(f \mid \pi) \\ &= \mathrm{vol}(Z_K)^{-1} \sum_{\pi \in \Pi_\chi(G)} m(\pi)\,\mathrm{tr}(f \mid \pi),\end{aligned} \tag{145}$$

where $f := f^p f_n f_\infty$ and the last equality follows from the assumption that K is small enough that $Z(\mathbb{Q})_K$ is trivial. Here we are denoted by χ the restriction to $A_{\mathbf{G}}(\mathbb{R})^0$ the of the central character of $\xi_{\mathbb{C}}^\vee$. Note that, by construction, f_∞ transforms under the center by the central character of $\xi_{\mathbb{C}}$ so, in particular, we see that $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$.

Let us now begin the proof of the result in earnest. Let us note that since $\mathsf{Sh}_K(\mathbf{G}, X)$ is proper for all neat compact open subgroups K of $\mathbf{G}(\mathbb{A}_f)$ we know from the proper base change theorem that an inclusion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ gives rise to an isomorphism

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) \xrightarrow{\approx} H^*(\mathsf{Sh}_{\mathbb{C}}, \mathcal{F}_\xi) \tag{146}$$

Moreover, by Artin's comparison theorem we obtain a natural isomorphism $\overline{\mathbb{Q}}_\ell$-spaces

$$H^*(\mathsf{Sh}_{\mathbb{C}}, \mathcal{F}_\xi) \xrightarrow{\approx} H^*_{\mathrm{sing}}(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi) \tag{147}$$

where we we imprecisely denoting by $H^*_{\mathrm{sing}}(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ the space

$$\varinjlim_K H^*_{\mathrm{sing}}(\mathsf{Sh}_K(G, X)^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_{\xi,K}) \tag{148}$$

which is in the Grothendieck group of $\overline{\mathbb{Q}_\ell}$-spaces.

Note that by Theorem II.3.12 this $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-module is semisimple. Thus, by definition, there exists a decomposition

$$H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f), \tag{149}$$

where π_f ranges over irreducible admissible $\mathbf{G}(\mathbb{A}_f)$-representations contained in $H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ and $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-space.

Let us note that since the $\mathbf{G}(\mathbb{A}_f)$-action on the tower Sh is defined $\mathbf{E}_{\boldsymbol{\mu}}$-rationally that the action of $\mathbf{G}(\mathbb{A}_f)$ and $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$ commute. For this reason, we see that the induced action of $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$ on $H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ induced from the above isomorphisms has the property that it preserves $\sigma(\pi_f)$, and thus we see that $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-representation (recalling our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$) of $W_{\mathbf{E}_{\boldsymbol{\mu}}}$.

Thus, in conclusion, pulling this decomposition back along the above isomorphisms we obtain a decomposition

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f) \tag{150}$$

where π_f travels over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ contained in $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ and $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-representation of $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$.

Remark II.5.3. Note that, *a priori*, the virtual $\overline{\mathbb{Q}_\ell}$-representation $\sigma(\pi_f)$ of Γ_E depends on the above chosen ambient identifications/data. But, as our description in Theorem II.2.1 shows, the traces of a dense set of elements of Γ_E are independent of these choices, and thus so is $\sigma(\pi_f)$.

We now fix for once and for all an admissible $\overline{\mathbb{Q}}_l$ representation π_f^0 of $\mathbf{G}(\mathbb{A}_f)$ satisfying the conditions of Theorem II.2.1. In particular, we assume there exists an automorphic $\overline{\mathbb{Q}}_l$-representation π of $\mathbf{G}(\mathbb{A})$ such that π is isomorphic to $\pi_f^0\pi_\infty$ where $\pi_\infty \in \Pi_\infty(\xi)$.

We now fix a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ satisfying the following properties.

- We assume that K is a neat subgroup,
- that $Z(\mathbb{Q})_K = 1$,
- and that π_f^K is nonempty.

We now fix f^∞ as in section II.3.2. Finally, we need to determine the cofinite set $S(\pi_f^0) \subset S(\mathbf{G})$ of theorem II.2.1. We define $S(\pi_f^0)$ so that for each $p \in S(\pi_f^0)$,

1. the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified,
2. we have a factorization $K = K^p K_{0,p}$ where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and $K_{0,p} \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup,
3. we can factor $f^\infty = f^p \mathbb{1}_{K_{0,p}}$ where $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p))$.

We briefly explain why the factorization in the third item can be made for all but finitely many p. We can write

$$f^\infty = \sum_i c_i \mathbb{1}_{K a_i K}, \tag{151}$$

where $c_i \in \mathbb{C}, a_i \in \mathbf{G}(\mathbb{A}_f)$. Now, for all but finitely many places, we have for all i, $(a_i)_p \in K_p$. Hence if S is the finite set of primes where this does not happen, we can write

$$f^\infty = (\sum_i c_i \mathbb{1}_{K_S (a_i)_S K_S}) \cdot \mathbb{1}_{K^S}, \tag{152}$$

which gives the desired factorization.

Now fix $p \in S(\pi_f^0)$ and a prime $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p. Now fix a $\tau \in W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$. We aim to describe $\mathrm{tr}(\tau \mid \sigma(\pi_f))$ as in theorem II.2.1. Note that since $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ is unramified at p (by smooth proper base change given the existence of smooth proper models by combinging [Kis10] and [Per12]) we may as well assume that $\tau = \Phi^j$ for some j where we denote by Φ the geometric Frobenius element of $W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$.

On the first hand, let us observe that we have the equality

$$\begin{aligned}
\mathrm{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) &= \mathrm{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_{\xi,K})\\
&= \sum_{\pi_f,\, \pi_f^K \neq 0} \mathrm{tr}(\Phi^j \times f^\infty \mid \pi_f^K \boxtimes \sigma(\pi_f))\\
&= \sum_{\pi_f,\, \pi_f^K \neq 0} \mathrm{tr}(f^\infty \mid \pi_f^K)\,\mathrm{tr}(\Phi^j \mid \sigma(\pi_f))\\
&= \mathrm{tr}(\Phi^j \mid \sigma(\pi_f^0))
\end{aligned} \tag{153}$$

where the last equality follows from the definition of f^∞.

On the other hand, by Equation (145), we have

$$\mathrm{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}})) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \mathrm{vol}(Z_K)^{-1} \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi)\,\mathrm{tr}(f \mid \pi), \tag{154}$$

where $f = f^p f_n f_\infty$.

Now by II.3.5, we can rewrite the right hand side of the above equation as

$$\mathrm{vol}(Z_K)^{-1} \sum_{\pi_f \in \Pi_{f,\chi}(\mathbf{G})} a(\pi_f)\,\mathrm{tr}(f^p f_n \mid \pi_f), \tag{155}$$

where $\Pi_{f,\chi}(\mathbf{G})$ denotes the set of admissible $\mathbf{G}(\mathbb{A}_f)$-representations π_f such that there exists a π_∞ an admissible $\mathbf{G}(\mathbb{R})$-representation such that $\pi_f \otimes \pi_\infty$ is an element of $\Pi_\chi(\mathbf{G})$.

At this point, we note that for any π_f, we have the equality

$$\begin{aligned}
\mathrm{tr}(f^p f_n \mid \pi_f) &= \mathrm{tr}(f^p \mid \pi_f^p)\,\mathrm{tr}(f_n \mid (\pi_f)_p) &(156)\\
&= \mathrm{tr}(f^p \mathbb{1}_{K_{0,p}} \mid \pi_f)\,\mathrm{tr}(f_n \mid (\pi_f)_p), &(157)
\end{aligned}$$

where the last step follows because $\mathrm{tr}(\mathbb{1}_{K_{0,p}} \mid (\pi_f)_p)$ equals 1 or 0 based on whether ${\pi_f}_p^{K_{0,p}}$ is nonempty or empty and in the latter case, we would also have $\mathrm{tr}(f_n \mid (\pi_f)_p) = 0$.

Now, by [Kot84a, Theorem 2.1.3], we have

$$\operatorname{tr}(f_n \mid (\pi_f)_p) = \operatorname{vol}(Z_K) \operatorname{tr}(\tau \mid r_{-\mu} \circ \varphi_{(\pi_f)_p}) p^{\frac{1}{2} j [\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}}:\mathbb{Q}_p] \dim \mathsf{Sh}}. \tag{158}$$

Finally, putting all the pieces together and recalling that $f^p \mathbb{1}_{K_{0,p}} = f^\infty$ which projects to the $(\pi_f^0)^K$-isotypic part of $H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_\xi)$, we get

$$\operatorname{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_\xi)) \tag{159}$$

is equal to

$$a(\pi_f) \operatorname{tr}(\tau \mid r_{-\mu} \circ \varphi_{(\pi_f)_p}) p^{\frac{1}{2} j [\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}}:\mathbb{Q}_p] \dim \mathsf{Sh}} \tag{160}$$

Combining this with Equation (153) proves Theorem II.2.1.

Part III

The unramified Scholze-Shin conjecture: the trivial endoscopic triple

III.1 Unramified unitary groups and their representations

In this section, we construct the various groups and representations that we use in the proof of our main result.

III.1.1 Local and global unitary groups

To begin, we fix a prime p of $\mathbb{Q}$ and a finite unramified extension $F/\mathbb{Q}_p$. We fix an isomorphism $\iota_p : \overline{\mathbb{Q}_p} \to \mathbb{C}$. Let E/F be the unique unramified extension of degree 2 and define $U_{E/F}(n)^*$ to be the unique up to isomorphism quasisplit unitary group of rank n over F for the extension E/F as in IV.4.15. Define G to be the group $\mathsf{Res}_{F/\mathbb{Q}_p} U_{E/F}(n)^*$. Note that G is unramified since $E/\mathbb{Q}_p$ is unramified. Note that $G_{\overline{\mathbb{Q}_p}}$ is isomorphic to a product of GL_n factors. We fix a nontrivial minuscule cocharacter μ of $G_{\overline{\mathbb{Q}_p}}$ by fixing a minuscule cocharacter μ_i of each factor the form

$$\mu_i(z) = \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \tag{161}$$

where the number of z factors and 1 factors in the above expression are a_i and b_i respectively. We assume that for at least one i, we have $a_i \notin \{0, n\}$.

Note that such a μ is minuscule but that not all minuscule μ are of this form. Since E is unramified over $\mathbb{Q}_p$, it is Galois and hence the reflex field of μ is a subfield of E which we denote E_μ.

We now note the following:

Lemma III.1.1. *There exists an extension of number fields* $\mathbf{E}/\mathbf{F}$ *satisfying the following properties:*

1. $\mathbf{E}_\mathfrak{q} = E$ *and* $\mathbf{F}_\mathfrak{p} = F$ *for some primes* $\mathfrak{q}$ *of* $\mathbf{E}$ *and* $\mathfrak{p}$ *of* F *such that* $\mathfrak{q} \cap \mathbf{F} = \mathfrak{p}$.
2. $\mathbf{F}$ *is totally real.*
3. $\mathbf{E}$ *is a quadratic imaginary extension of* $\mathbf{F}$.
4. $\mathbf{F} \neq \mathbb{Q}$.

Proof. The construction of $\mathbf{F}$ follows from [Art13b, Lemma 6.2.1] by taking any $r_0 > 1$. Indeed, the construction of loc. cit. produces $\mathbf{F}$ satisfying the desired conditions of 1. and 2. and the existence of more than one real place on $\mathbf{F}$ implies condition 3.

We argue about the existence of $\mathbf{E}$ similarly. Indeed, the only assumption for which the arguments of loc. cit. don't apply directly to is the assumption that $\mathbf{E}/\mathbf{F}$ is imaginary. But, this follows immediately from the method of loc. cit. since for an embedding of $F^2 \hookrightarrow \mathbb{R}^2$ the monic polynomials with imaginary roots is open since it corresponds to $(b, c) \in \mathbb{R}^2$ such that $b^2 - 4c < 0$. $\square$

We now define $\mathbf{U}^*$ to be the group $U_{\mathbf{E}/\mathbf{F}}(n)^*$ and $\mathbf{G}^*$ to be $\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}^*$. The previously defined ι_p induces an isomorphism $\mathbf{G}^*_{\overline{\mathbb{Q}_p}} \cong \mathbf{G}^*_{\mathbb{C}}$. Note that G is a direct factor of $\mathbf{G}^*_{\mathbb{Q}_p}$ and hence again by ι_p, we get that $G_{\mathbb{C}}$ is a direct factor of $\mathbf{G}^*_{\mathbb{C}}$. Define a minuscule cocharacter $\boldsymbol{\mu}$ of $\mathbf{G}^*$ so that $\boldsymbol{\mu}$ restricts to μ on the $G_{\mathbb{C}}$ factor and is trivial elsewhere.

We would now like to record the existence of a certain unitary group over a global field.

Proposition III.1.2. *There exists an inner form $\mathbf{U}$ of $\mathbf{U}^*$ and hence an inner form $\mathbf{G} := \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}$ of $\mathbf{G}^*$ such that:*

1. *The group $\mathbf{G}^{\mathrm{ad}}$ is $\mathbf{F}$-anisotropic.*
2. *The group $\mathbf{G}$ has no relevant global endoscopy.*
3. *The group G is a direct factor of $\mathbf{G}_{\mathbb{Q}_p}$.*
4. *Let $\{v\}$ denote the infinite places of $\mathbf{F}$. Given any set $\{(p_v, q_v)\}$ of pairs of non-negative integers such that $p_v + q_v = n$ we have that $U_v \cong U(p_v, q_v)$.*

Proof. We shall use the terminology as in Lemma IV.4.25. In particular, we shall construct $\mathbf{U}$ by constructing $\mathbf{U}_v \in \mathsf{InnForm}(\mathbf{U}^*_v)$ for all places v of $\mathbf{F}$. Begin by setting $\mathbf{U}_v := U(p_v, q_v)$ for each $v \mid \infty$ as in condition 4. of the proposition. Let us also set $\mathbf{U}_{v_0} := \mathbf{U}^*_{v_0}$ where $v_0 = \mathfrak{p}$ is the prime from Lemma III.1.1. Choose some finite place v_0' of F different than v_0 and set $\mathbf{U}_{v_0'} := D^\times_{\frac{1}{n}}$. Let us set

$$\epsilon := \sum_{v|\infty} \epsilon_v(\mathbf{U}_v) + \epsilon_{v_0}(\mathbf{U}_{v_0}) + \epsilon_{v_0'}(\mathbf{U}_{v_0'}) \tag{162}$$

This is an element of $\mathbb{Z}/2\mathbb{Z}$. If $\epsilon = 0$ let us set $\mathbf{U}_v := \mathbf{U}^*_v$ for all $v \nmid \infty$ such that $v \notin \{v_0, v_0'\}$. If $\epsilon \neq 0$ then necessarily n is even. In this case choose some finite split (relative to E) place v_0'' and set $\mathbf{U}_{v_0''} := D^\times_{\frac{n-1}{n}}$ and then set $\mathbf{U}_v := \mathbf{U}^*_v$ for $v \nmid \infty$ such that $v \notin \{v_0, v_0', v_0''\}$. By construction we have that $\sum_v \epsilon_v(\mathbf{U}_v) = 0$ and thus by Lemma IV.4.25 there exists a unique $\mathbf{U} \in \mathsf{InnForm}(\mathbf{U}^*)$ such that $\mathbf{U}_{\mathbb{Q}_v} \cong \mathbf{U}_v$.

Note that

$$\mathbf{G}_{\mathbb{Q}_p} \cong \prod_{v|p} \mathsf{Res}_{\mathbf{F}_v/\mathbb{Q}_p} \mathbf{U}_v \tag{163}$$

and thus by construction we see that $\mathbf{G}_{\mathbb{Q}_p}$ contains as a factor $\mathsf{Res}_{\mathbf{F}_{v_0}/\mathbb{Q}_p} \mathbf{U}_v$. But, by construction, $\mathbf{U}_v \cong U_{E/F}(n)^*$ and $\mathbf{F}_{v_0} \cong F$ and thus condition 3. is automatically satisfied. Also, evidently condition 4. is satisfied. Thus, it remains to show that conditions 1. and 2. are satisfied

Now, note that since $\mathbf{U}$ is an element of $\mathsf{InnForm}(U_{\mathbf{E}/\mathbf{F}}(n^*))$ we know by Lemma IV.4.16 that $U \cong U(\Delta, *)$ where Δ is some central simple $\mathbf{E}$-algebra. Combining Lemma IV.4.19 and Lemma IV.4.27 it suffices to show that Δ must be a division algebra. To do this, note that by Lemma IV.4.13 one has an isomorphism $\mathbf{U}_E \cong \Delta^\times$. By 1. of Lemma IV.4.19 it suffices to show that $U(E)$ contains no non-trivial unipotent elements. But, $\mathbf{U}(E) \subseteq U_{v_0'}(E_{v_0'})$. Note though that we have an isomorphism $(U_{v_0'})_{E_{v_0'}} \cong (D^\times_{\frac{1}{n}})^2$ and since $(D^\times_{\frac{1}{n}})^2$ is anisotropic modulo center we see that this contains no non-trivial unipotent elements as desired. □

We now fix global groups $\mathbf{U}$ and $\mathbf{G}$ satisfying the statement of III.1.2 where we fix the set $\{(p_v, q_v)\}$ so that $p_v = a_v$ and $q_v = b_v$ where we recall that $\{(a_v, b_v)\}$ comes from the definition of $\boldsymbol{\mu}$. We get a conjugacy class of cocharacters of $\mathbf{G}$ associated to $\boldsymbol{\mu}$. We denote the reflex field of this conjugacy class by $\mathbf{E}_{\boldsymbol{\mu}}$. In the present case, $\mathbf{E}$ and $\mathbf{F}$ are not assumed to be Galois. Hence it need not be true that $\mathbf{F} \subset \mathbf{E}_{\boldsymbol{\mu}}$. All we can say is that $\mathbf{E}_{\boldsymbol{\mu}}$ is a subfield of the Galois closure, $c(\mathbf{E})$ of $\mathbf{E}$. Since we have fixed the isomorphism $\iota_p : \overline{\mathbb{Q}_p} \to \mathbb{C}$, we get a cocharacter of $\mathbf{G}_{\mathbb{Q}_p}$ which we also call $\boldsymbol{\mu}$. On the one hand, the reflex field of this $\boldsymbol{\mu}$ is given by the completion of $\mathbf{E}_{\boldsymbol{\mu}}$ at the place $\mathfrak{p}$ over p corresponding to ι_p. On the other hand, by construction, $\mathbf{G}_{\mathbb{Q}_p} = G \times G'$ and hence $\boldsymbol{\mu} = (\mu, \mu')$ where μ is fixed before and μ' is trivial. Hence the reflex field of $\boldsymbol{\mu}$ in $\mathbf{G}_{\mathbb{Q}_p}$ is E_μ. Thus, we have shown that if $\mathfrak{p}$ is the place of $\mathbf{E}_{\boldsymbol{\mu}}$ determined by ι_p, then $\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}} = E_\mu$.

III.1.2 Shimura data for unitary groups

In this section we will write down the general conditions necessary to have a Shimura datum of the form $(\mathbf{G}, X)$ where $\mathbf{G} = \mathsf{Res}_{\mathbf{F}/\mathbb{Q}} \mathbf{U}$ and where $\mathbf{F}$ is some number field, $\mathbf{E}$ is a quadratic extension, and $\mathbf{U}$ is an inner form of $U_{E/F}(n)^*$ for some n. We will then, in particular, verify that we can find a Shimura datum of abelian type $(\mathbf{G}, X)$ where $\mathbf{G}$ is as in §III.1.1. See [RSZ17, §3] for an alternative discussion of the following.

Let us begin by saying that $\mathbf{U}$ (or $\mathbf{G}$) is of *non-compact type* if for some infinite place v of $\mathbf{F}$ we have that $\mathbf{U}_{\mathbf{F}_v}$ is not $\mathbb{R}$-anisotropic. In other words, $\mathbf{G}$ is of compact type if $\mathbf{G}(\mathbb{R})$ is compact, and being of non-compact type just means that it is not of compact type. We then have the following claim:

Lemma III.1.3. *Suppose that E is a CM field and $\mathbf{G}$ is of non-compact type. Then, there is a Shimura datum $(\mathbf{G}, X)$ of abelian type.*

Proof. So, let us assume that $\mathbf{U}$. Let

$$h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \cong \prod_i U(p_i, q_i) \tag{164}$$

(where we have a priori fixed this latter isomorphism) be defined in terms of its projections h_i defined as follows. If $p_i = 0$ or $q_i = 0$ we define h_i to be trivial. Otherwise, define h_i as follows:

$$h_i(z) := \begin{pmatrix} \frac{z}{\bar{z}} & & & & & \\ & \ddots & & & & \\ & & \frac{z}{\bar{z}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \tag{165}$$

where there are p_i entries of $\frac{z}{\bar{z}}$ and q_i entries of 1. Set X to be the $\mathbf{G}(\mathbb{R})$-conjugacy class of h. We claim that $(\mathbf{G}, X)$ is a Shimura datum of abelian type.

The fact that $(\mathbf{G}, X)$ is a Shimura datum is elementary and left to the reader (the assumption that $\mathbf{U}$ is of non-compact type being used in Axiom SV3 of [Mil04]). To see that it's of abelian type, we must find an associated Hodge type datum. Let $G\mathbf{U}$ denote the associated unitary similitude group associated to $\mathbf{U}$ and set $\mathbf{H} := \mathsf{Res}_{\mathbf{F}/\mathbb{Q}} G\mathbf{U}$. We then define $\mathbf{H}^{\mathbb{Q}}$ to be the fiber product $\mathbf{H} \times_{\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_{m,\mathbb{Q}}$ where the map $\mathbf{H} \to \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,\mathbf{F}}$ is the similitude character and the map $\mathbb{G}_{m,\mathbb{Q}} \to \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,\mathbf{F}}$ is the usual inclusion. We define a morphism

$$h' : \mathbb{S} \to (\mathbf{H}^{\mathbb{Q}})_{\mathbb{R}} \tag{166}$$

as follows. Begin by noting that

$$(\mathbf{H}^{\mathbb{Q}})_{\mathbb{R}} = \left\{ (g_i) \in \prod_i GU(p_i, q_i)\rangle) : c(g_i) = c(g_j) \text{ for all } i, j \text{ and } c(g_i) \in \mathbb{R}^{\times} \right\} \tag{167}$$

Let us fix one such isomorphism. We then define h', via this fixed isomorphism, by its projections h'_i to each $GU(p_i, q_i)$ by

$$h'_i(z) := \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & \overline{z} & & \\ & & & & \ddots & \\ & & & & & \overline{z} \end{pmatrix} \tag{168}$$

where there are p_i copies of z, and q_i copies of $\overline{z}$. One can then check that $(\mathbf{H}^{\mathbb{Q}}, h')$ defines a PEL type Shimura datum (e.g. see [Mil04, Chapter 8]).

Note now that $(\mathbf{H}^{\mathbb{Q}})^{\mathrm{der}}$ is naturally isomorphic to $\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}^{\mathrm{der}}$ which is, likewise, equal to $\mathbf{G}^{\mathrm{der}}$. Let $(\mathbf{H}^{\mathbb{Q}})^{\mathrm{der}} \to \mathbf{G}^{\mathrm{der}}$ be the identity map. It's not hard to see then that this induces an isomorphism of Shimura datum between $((\mathbf{H}^{\mathbb{Q}})^{\mathrm{ad}}, (h')^{\mathrm{ad}})$ and $(\mathbf{G}^{\mathrm{ad}}, h^{\mathrm{ad}})$. Thus, $(\mathbf{G}, X)$ is of abelian type. □

We now observe that $\mathbf{G}$ as in §III.1.1 is of non-compact type since μ and hence $\boldsymbol{\mu}$ is non-trivial. We can define a Shimura datum $(\mathbf{G}, X)$ as in the previous lemma. In particular, we note that by construction, the conjugacy class of cocharacters of $\mathbf{G}_{\mathbb{C}}$ associated to X contains $\boldsymbol{\mu}$ as an element.

III.1.3 Local and global representations

We now fix a square integrable irreducible admissible representation $\pi_p^0 \in \mathbb{C}[G(\mathbb{Q}_p)]$. We also fix a Shimura datum $(\mathbf{G}, X)$ as in the last section, as well as an algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of $\mathbf{G}$ with regular highest weight. We have by assumption $\mathbf{G}_{\mathbb{Q}_p} = G \times G'$. Fix a square integrable representation π'_p of $G'(\mathbb{Q}_p)$ so that $\pi_p^0 \boxtimes \pi'_p$ is a square-integrable representation of $\mathbf{G}(\mathbb{Q}_p)$.

We need the following proposition

Proposition III.1.4. *There exists a representation π of $\mathbf{G}(\mathbb{A})$ such that π_f appears $H^*(\mathsf{Sh}(\mathbf{G}, X), \mathcal{F}_\xi)$ and such that $\pi_p \cong \pi_p^0 \boxtimes \pi'_p$.*

Proof. This is an easy consequence of [Shi12, Theorem 5.7]. We set S to be the places of $\mathbb{Q}$ where $\mathbf{G}$ is ramified plus the place p. Then we fix a square integrable representation π_S of $\mathbf{G}(\mathbb{Q}_S)$ such that $(\pi_S)_p = \pi_p^0 \boxtimes \pi'_p$. We let $\widehat{U}$ be the $\widehat{\mu}^{pl}$-regular set equal to the orbit $\mathcal{O}$ of the unramified unitary characters of $\mathbf{G}(\mathbb{Q}_S)$ acting on π_S as in [Shi12, Example 5.6]. We note that at p, we have that any $\pi'_S \in \widehat{U}$ satisfies $(\pi'_S)_p = \pi_p^0 \boxtimes \pi'_p$ since $\mathbf{G}(\mathbb{Q}_p)$ has no split torus in its center. We then apply Theorem 5.7 of Shin's paper to get the desired result. Note in particular, that π_f appears in $H^*(\mathsf{Sh}(\mathbf{G}, X), \mathcal{F}_\xi)$ since it is ξ-cohomological at ∞. □

We now fix a global π satisfying the properties of the above theorem. Note that since we have assumed ξ has regular highest weight, it follows from the remark after Theorem 1 of [Kot92a] that π is discrete and hence elliptic at infinity.

III.2 Construction of the global Galois representation

We continue with the notation fixed as in III.1. In this section only, we use the Galois form of L-groups. We do so because we work primarily with Galois representations instead of A-parameters.

III.2.1 Unitary shimura varieties

We first define a morphism of L-groups

$$\lambda : {}^L\mathbf{G} \to {}^L\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n. \tag{169}$$

As a group, $\widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}}_n$ is isomorphic to

$$\left(\prod_{\Gamma_\mathbb{Q}/\Gamma_\mathbf{E}} \mathrm{GL}_n(\mathbb{C})\right) \rtimes \Gamma_\mathbb{Q}. \tag{170}$$

We fix a subset $X \subset \Gamma_\mathbb{Q}/\Gamma_\mathbf{E}$ such that the map

$$\Gamma_\mathbb{Q}/\Gamma_\mathbf{E} \to \Gamma_\mathbb{Q}/\Gamma_\mathbf{F}, \tag{171}$$

induces a bijection

$$X \xrightarrow{\approx} \Gamma_\mathbb{Q}/\Gamma_\mathbf{F}. \tag{172}$$

We define $X^\perp := \Gamma_\mathbb{Q}/\Gamma_\mathbf{E} \setminus X$. We now construct λ by

$$\lambda(g_1, ..., g_m, w) = (g_1, ..., g_m, J_N(g_1^{-1})^t J_N^{-1}, ..., J_N(g_m^{-1})^t J_N^{-1}, w), \tag{173}$$

where the left hand side is an element of

$$(\prod_{\Gamma_\mathbb{Q}/\Gamma_\mathbf{F}} \mathrm{GL}_n(\mathbb{C})) \rtimes \Gamma_\mathbb{Q} = {}^L\mathbf{G}, \tag{174}$$

and the right hand side is an element of

$$(\prod_X \mathrm{GL}_n(\mathbb{C}) \times \prod_{X^\perp} \mathrm{GL}_n(\mathbb{C})) \rtimes \Gamma_\mathbb{Q} = {}^L\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n(\mathbb{C}). \tag{175}$$

III.2.2 The identification of $\sigma(\pi_f)$

We continue with notation as in III.1. In particular, $(\mathbf{G}, X)$ is an abelian type Shimura datum, ξ is an irreducible algebraic representation of $\mathbf{G}_{\mathbb{C}}$, and π is an irreducible automorphic representation of $\mathbf{G}(\mathbb{A})$ that is ξ-cohomological at ∞. By IV.3.1, we get an irreducible discrete automorphic representation $\mathrm{BC}(\pi)$ of $GL_n(\mathbb{A}_{\mathbf{E}})$ that is conjugate self-dual with infinitesimal character $(\xi\otimes\xi)^\vee$. Note that since ξ is regular, that $(\xi\otimes\xi)^\vee$ is slightly regular so that we can apply [Shi11, Theorem 1.2].

We now apply [Shi11, Theorem 1.2] to get a representation $\sigma(\mathrm{BC}(\pi))$ of $\Gamma_{\mathbf{E}}$ with coefficients in $\overline{\mathbb{Q}_\ell}$. In this section, we identify an explicit relationship of the Galois representation $\sigma(\pi_f)$, as in Theorem II.2.1, and the representation $\sigma(\mathrm{BC}(\pi))$ of $GL_n(\overline{\mathbb{Q}_\ell})$, as in [Shi11, Theorem 1.2].

Now consider the representation

$$\sigma := \iota_\ell \sigma(\mathrm{BC}(\pi)) : \Gamma_{\mathbf{E}} \to \mathrm{GL}_n(\mathbb{C}). \tag{176}$$

We identify $GL_n(\mathbb{C})$ with $\widehat{\mathrm{GL}_{n\mathbf{E}}} \subset {}^L GL_{n\mathbf{E}}$ and consider the equivalence class $[\sigma]$ up to conjugacy by an element of $\widehat{GL_{n\mathbf{E}}}$. Thus, we have $[\sigma] \in H^1(\Gamma_E, \widehat{GL_{n\mathbf{E}}})$. Now, by a variant of Shapiro's lemma, [Bor79, Lemma 4.5], we get a class of $H^1(\Gamma_{\mathbb{Q}}, \widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_{n\mathbf{E}}})$. Pick a representative ρ of this class. Then again by [Bor79, Lemma 4.5], we have that the projection of ρ to the factor corresponding to the trivial coset of $\Gamma_{\mathbf{E}}$ is a representative of $[\sigma]$.

We need a few lemmas.

Lemma III.2.1. *Let E/F be an unramified extension of p-adic local fields. Let H be an unramified reductive group over E. Fix a hyperspecial subgroup $K = \mathcal{H}(\mathcal{O}_E) \subset H(E)$ and let π be an irreducible admissible representation of $H(E)$ unramified with respect to K. Then since $H(E) = (\mathsf{Res}_{E/F}H)(F)$, we can also naturally consider π to be an unramified representation of $(\mathsf{Res}_{E/F}H)(F)$ with respect to $(\mathsf{Res}_{\mathcal{O}_E/\mathcal{O}_F}\mathcal{H})(\mathcal{O}_F)$. We denote this representation by π'.*

Now, let $\psi_\pi = \mathsf{LL}_E(\pi)$ and $\mathcal{I}\psi_\pi$ be the equivalence class of parameters of $\mathsf{Res}_{E/F}H$ coming from ψ_π by Shapiro's lemma. Then $\mathcal{I}\psi_\pi = LL_F(\pi')$.

Proof. (Sketch) Let us note that since H is unramified it has an unramified maximal torus. Indeed, let $\mathcal{H}$ be a reductive model for H over $\mathcal{O}_E$. Note that the variety of maximal tori X is smooth over $\mathcal{O}_E$ (e.g. see [Con14, Theorem 3.2.6]) we can use Hensel's lemma to lift a maximal torus of $\mathcal{H}_k$ (where k is the residue field) to a maximal torus of $\mathcal{H}$ whose generic fiber is an unramified torus of H. Note then that by the argument in [BR94, §1.12] we can then reduce the argument to that of tori. This is then a well-known result (e.g. see [L+97]). □

We now return to the notation before the previous lemma.

Lemma III.2.2. *For each place p of $\mathbb{Q}$ such that* $\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n, \mathbf{E}$ *and* $\mathrm{BC}(\pi)$ *are unramified at p, we have* $\rho|_{\Gamma_{\mathbb{Q}_p}} = LL_{\mathbb{Q}_p}(\mathrm{BC}(\pi)_p)$.

Proof. We consider the following diagram

$$\begin{tikzcd} H^1(\mathbf{E}, \widehat{\mathrm{GL}_n, \mathbf{E}}) \arrow[r] \arrow[d, leftrightarrow] & \prod_{\mathfrak{p}|p} H^1(\mathbf{E}_{\mathfrak{p}}, \widehat{\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}) \arrow[d, leftrightarrow] \\ H^1(\mathbb{Q}, \widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n, \mathbf{E}}) \arrow[r] & \prod_{\mathfrak{p}|p} H^1(\mathbb{Q}_p, \widehat{\mathsf{Res}_{\mathbf{E}_{\mathfrak{p}}/\mathbb{Q}_p}\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}), \end{tikzcd} \tag{177}$$

where the vertical arrows are Shapiro isomorphisms, the top horizontal arrow is a product of restriction maps to each $\Gamma_{\mathbf{E}_{\mathfrak{p}}}$, and the bottom horizontal map is the composition of the restriction to $\Gamma_{\mathbb{Q}_p}$ and the isomorphism

$$H^1(\mathbb{Q}_p, (\widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n, \mathbf{E}})_{\mathbb{Q}_p}) \cong \prod_{\mathfrak{p}|p} H^1(\mathbb{Q}_p, \widehat{\mathsf{Res}_{\mathbf{E}_{\mathfrak{p}}/\mathbb{Q}_p}\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}). \tag{178}$$

We claim that this diagram commutes. Indeed the vertical maps are just projections onto the identity coset factors and the horizontal maps are products of restrictions.

But now, we have from [Shi11, Thm 1.2] that $\sigma|_{\mathbf{E}_{\mathfrak{p}}} = \mathsf{LL}_{\mathbf{E}_{\mathfrak{p}}}(\mathrm{BC}(\pi)_{\mathfrak{p}})$. Then by commutativity of the above diagram and the previous lemma we get the desired result. □

We now take the dominant cocharacter $\boldsymbol{\mu}$ of $\mathbf{G}_{\mathbb{C}} \cong \prod_{\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}} (\mathrm{GL}_n)_{\mathbb{C}}$ associated to the Shimura datum $(\mathbf{G}, X)$ and write it as a product of cocharacters $(\boldsymbol{\mu}_{\tau_1}, ..., \boldsymbol{\mu}_{\tau_m})$ where τ ranges over $\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}$. We then define the cocharacter $(-\boldsymbol{\mu}, 0)$ of

$$(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n)_{\mathbb{C}} = (\prod_X \mathrm{GL}_n(\mathbb{C}) \times \prod_{X^\perp} \mathrm{GL}_n(\mathbb{C})) \tag{179}$$

so that the character is $-\boldsymbol{\mu} = (-\boldsymbol{\mu}_{\tau_1}, ... - \boldsymbol{\mu}_{\tau_m})$ on the copies of GL_n indexed by X and 0 on the copies of GL_n indexed by $X^\perp$. We denote the reflex field of $(\boldsymbol{\mu}, 0)$ by $\mathbf{E}_{(\boldsymbol{\mu},0)}$. Then using ι_p, we consider $(\boldsymbol{\mu}, 0)$ as a cocharacter of $(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n)_{\mathbb{Q}_p}$ and observe that the localization of $\mathbf{E}_{(\boldsymbol{\mu},0)}$ at the place $\mathfrak{p}$ equals $E_{(\mu,0)}$ and moreover we have the following observation:

Lemma III.2.3. *We have an equality of fields* $(\mathbf{E}_{\boldsymbol{\mu}})_{\mathfrak{p}} = (\mathbf{E}_{(\boldsymbol{\mu},0)})_{\mathfrak{p}}$.

Proof. Let us note that it suffices to show that the reflex fields of the local cocharacters $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, 0)$ agree. To do this let us note that we have a natural embedding of $\mathbb{Q}$-groups

$$\mathbf{G} \hookrightarrow \mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_{n,\mathbf{E}} \tag{180}$$

Upon base changing this to $\overline{\mathbb{Q}}$ we obtain a Galois invariant embedding

$$\mathbf{G}_{\overline{\mathbb{Q}}} \hookrightarrow (\mathsf{Res}_{E/\mathbb{Q}}\mathrm{GL}_{n,E})_{\overline{\mathbb{Q}}} \cong \prod_X \mathrm{GL}_{n,\overline{\mathbb{Q}}} \times \prod_{X^\perp} \mathrm{GL}_{n,\overline{\mathbb{Q}}} \tag{181}$$

with notation as above. In particular, we see that we get a natural $\Gamma_{\mathbb{Q}_p}$-equivariant embedding

$$\mathbf{G}_{\overline{\mathbb{Q}_p}} \hookrightarrow \prod_X \mathrm{GL}_{n,\overline{\mathbb{Q}_p}} \times \prod_{X^\perp} \mathrm{GL}_{n,\overline{\mathbb{Q}_p}} \tag{182}$$

Note that this map sends $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ to $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}J_N^{-1})$. It is fairly evident then that the reflex fields of $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}J_N^{-1})$ are equal. Indeed, only non-trivial factors of $\boldsymbol{\mu}$ correspond to elements of X coming from $\mathsf{Res}_{F_v/\mathbb{Q}_p}U_{E_w/F_v}$, and $E_w/\mathbb{Q}_p$ is Galois and so the only relevant part of the Galois action on the right hand side of Equation (182) act by the transposition interchanging X and $X^\perp$ and then by the natural action of $\mathrm{Gal}(F_v/\mathbb{Q}_p)$. Finally, one sees that $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, 0)$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}J_N^{-1})$ have the same reflex field since $J_N\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}J_N^{-1}$ is never conjugate to $\boldsymbol{\mu}$ by our assumption that $\boldsymbol{\mu}$ is non-trivial. The conclusion follows. $\square$

Let $\mathbf{E}^*$ be the compositum of $\mathbf{E}_{\boldsymbol{\mu}}$ and $\mathbf{E}_{(\boldsymbol{\mu},0)}$. We have $\mathbf{E}^*_{\mathfrak{p}} = E_\mu$. We then get a representation

$$r_{(-\boldsymbol{\mu},0)} : {}^L(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n)|_{\Gamma_{\mathbf{E}_{(\boldsymbol{\mu},0)}}} \to \mathrm{GL}_N(\mathbb{C}), \tag{183}$$

as described in the notation at the beginning of the paper. We record the following lemma.

Lemma III.2.4. *Take $\lambda : {}^L\mathbf{G} \to {}^L\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n$ as in (169). Then we have have an equality restricted to $\widehat{\mathbf{G}} \rtimes \Gamma_{\mathbf{E}^*}$.*

$$r_{(-\boldsymbol{\mu},0)} \circ \lambda = r_{-\boldsymbol{\mu}}. \tag{184}$$

Proof. This follows more or less immediately from the definition of λ. $\square$

We then have the following proposition:

Proposition III.2.5. *Let q be an element of $S(\pi_f)$ and $\mathfrak{q}$ any place of $\mathbf{E}^*$ lying over q. Then, we have an equality*

$$\mathrm{tr}\left(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}\right) = \mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{-\boldsymbol{\mu}} \circ \mathsf{LL}_{\mathbb{Q}_q}(\pi_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{185}$$

Before giving the proof of the above proposition, we record the following corollary, which is the key result of the section.

Corollary III.2.6. *For each $q \in S(\pi_f)$ and each place $\mathfrak{q}$ of $\mathbf{E}^*$ lying over q, we have the following equality*

$$a(\pi_f)\operatorname{tr}(\Phi_{\mathfrak{q}}|(r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) = \operatorname{tr}(\Phi_{\mathfrak{q}}|\sigma(\pi_f)). \tag{186}$$

In particular, it follows that we have the following equality in the Grothendieck group of $W_{\mathbf{E}^*}$-representations

$$a(\pi_f)[(r_{(-\boldsymbol{\mu},0)} \circ \rho) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}] = \sigma(\pi_f), \tag{187}$$

and hence by [Shi11, Thm 1.2], for any (not just unramified) prime q of $\mathbb{Q}$ and each place $\mathfrak{q}$ of $\mathbf{E}^*$ over q, and for $\tau \in W_{\mathbf{E}^*_{\mathfrak{q}}}$,

$$a(\pi_f)\operatorname{tr}(\tau|(r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{W_{\mathbf{E}^*_{\mathfrak{q}}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) = \operatorname{tr}(\tau|\sigma(\pi_f)). \tag{188}$$

In particular, we will want to apply this corollary to the chosen prime p and the place $\mathfrak{p}$ of $\mathbf{E}^*$ coming from ι_p.

Proof. (Proposition III.2.5) By III.2.2 and since $\Phi_{\mathfrak{q}} \in \Gamma_{\mathbb{Q}_q}$, we have

$$\operatorname{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \operatorname{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}(\pi)_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{189}$$

Now, by IV.3.1, the above equals

$$\operatorname{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}_q(\pi_q))|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{190}$$

By the compatibility of local base change with the unramified local Langlands correspondence [Mın11, Thm 4.1], we then have

$$\operatorname{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}_q(\pi_q))|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \operatorname{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ \mathsf{LL}_{\mathbb{Q}_q}(\pi_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{191}$$

Finally, by III.2.4, we get

$$\operatorname{tr}(\Phi_{\mathfrak{p}} \mid r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ \mathsf{LL}_{\mathbb{Q}_p}(\pi_p)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \operatorname{tr}(\Phi_{\mathfrak{p}} \mid r_{-\boldsymbol{\mu}} \circ \mathsf{LL}_{\mathbb{Q}_p}(\pi_p)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) \tag{192}$$

□

III.3 Traces at a place of bad reduction and pseudo-stabilization

In this section we record an analogue of the trace formula as in §II.4, as well as the pseudo-stabilization of that formula as in §II.5. In particular, we keep the notation and assumptions the same as in §II.4 throughout this section with one exception. Namely, we fix a compact open subgroup $K_p \subseteq K_{0,p}$ and then set $K := K^p K_p$.

The first main result is the following:

Theorem III.3.1 ([You19, Theorem 4.4.1]). *Let $h \in \mathscr{H}_{\mathbb{Q}}(\mathcal{G}(\mathbb{Z}_p), K_p)$ and let $\tau \in W_{E_{\mathfrak{p}}}$. Then, there exists a a function $\phi_{\tau,h} \in \mathscr{H}_{\mathbb{Q}}(\mathbf{G}((E_{\mathfrak{p}})_j))$ (independent of the choice of ℓ) such that for any $f^p \in \mathscr{H}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G}(\mathbb{A}_f^p), K^p)$ the following equality holds*

$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \sum_{\substack{\mathfrak{t}=(\gamma_0,\gamma,\delta)\\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\mathfrak{t}) O_\gamma(f^p) TO_\delta(\phi_{\tau,h}) \operatorname{tr}\xi(\gamma_0) \quad (193)$$

The proof of the above, or rather the simplifications to the formula made in [You19, Theorem 4.4.1], are the same as in the proof of Theorem II.4.2.

Let us now fix a function $f^p \in \mathscr{H}_{\mathbb{Q}}(\mathbf{G}(\mathbb{A}_f^p), K^p)$ with the property $f^p \mathbb{1}_{K_p}$ is a projector from $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ on to $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)[(\pi_f)^{K^p}]$ and let f_∞ be as in §II.3.1. Let us also set $f_{\tau,h} \in \mathscr{H}(\mathbf{G}(\mathbb{Q}_p))$ to be a transfer of $\phi_{\tau,h}$ (which exists by the results of [Wal08]).

We then have the following claim:

Proposition III.3.2. *The following equality holds:*

$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \tau_K(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\text{s.s.}}} SO_\gamma(f^p f_{\tau,h} f_\infty) \qquad (194)$$

Proof. The proof of this result is exactly the same as in the proof of Theorem II.5.1. The only substantive change is that the proof of the analogue of (141) is now by the twisted fundamental lemma (as in [Wal08]). □

We then deduce that

$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f^p f_{\tau,h} f_\infty) \qquad (195)$$

Note then that we can rewrite the right-hand side of this equation as

$$\sum_{\pi_f \in \Pi_{f,\chi}(\mathbf{G})} a(\pi_f) \operatorname{tr}(f^p f_{\tau,h} \mid \pi_f) = \sum_{\pi_f \in \Pi_{f,\chi}(\mathbf{G})} a(\pi_f) \operatorname{tr}(f^p \mathbb{1}_{K_p} \mid \pi_f) \operatorname{tr}(f_{\tau,h} \mid \pi_p) \qquad (196)$$

Note though that by construction $a(\pi_f)\operatorname{tr}(f^p \mathbb{1}_{K_p})$ will vanish unless $(\pi_f)^K$ has non-trivial isotypic component in $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)$ and the away-from-p component of π_f agrees with that of $\pi_{0,f}^p$. Let us call this set S.

From this, we see that our sum reduces to

$$\sum_{\pi_f \in S} a(\pi_f) \operatorname{tr}(f_{\tau,h} \mid \pi_{f,p}) \qquad (197)$$

Note though that we have the following reuslt:

Lemma III.3.3. *The set S is precisely $\Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \xi_p)$ where ψ_p is the A-parameter associated to $\pi_{0,p}$.*

Proof. Let us denote by S' the set of $\mathbf{G}(\mathbb{Q}_p)$-components of the irreducible factors of $L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\pi^p]$. By Matushima's formula it is clear that $S \subseteq S'$. Moreover, by Lemma I.6.3 we know that S' is precisely $\Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \omega)$. Thus, it suffices to show that $S = S'$.

Equivalently, by Corollary II.3.9, for every $\pi_p \in S'$ we need to show that $a(\pi_p \otimes \pi_f^p) \neq 0$. But, since ξ is regular we see by Theorem II.3.10 that $a(\pi_p \otimes \pi_f^p) \neq 0$ if and only if $m(\pi_p \otimes \pi^p) \neq 0$. This is precisely the claim that $S = S'$. $\square$

From the above, we deduce the following:

Proposition III.3.4.

$$\mathrm{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = a(\pi_f) \sum_{\pi_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \xi_p)} \mathrm{tr}(f_{\tau,h} \mid \pi_p) \tag{198}$$

III.4 The Scholze-Shin conjecture in certain unramified cases

In this section we prove our main result of the paper. Let E/F and G be as in §III.1.1 and π_p^0 a square integrable representation of $G(\mathbb{Q}_p)$ and $\pi_p^0 \boxtimes \pi_p'$ an irreducible square integrable representation of $\mathbf{G}(\mathbb{Q}_p)$ as in §III.1.3. Let ψ_p and ψ_p' be the Arthur parameters associated to π_p^0 and π_p' respectively as in [KMSW14, Theorem 1.6.1]. In particular, $\pi_p^0 \boxtimes \pi_p'$ has Arthur parameter $\psi_p \oplus \psi_p'$. Since π_p^0 and π_p' are tempered, ψ_p and ψ_p' are also bounded Langlands parameters. Let $(\mathbf{G}, X)$ be as in §III.1.2 and let $\boldsymbol{\mu}$ and μ be as in §III.1.1.

We now prove the following which is a special case of the Scholze-Shin conjecture [SS13, Conj 7.1].

Theorem III.4.1. *Pick any natural number $j \geq 1$ and $\tau \in \mathrm{Frob}^j I_{E_\mu} \subset W_{E_\mu}$. Pick $h \in \mathscr{H}(G(\mathbb{Z}_p))$. Then*

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi_p|_{W_{E_\mu}} \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p). \tag{199}$$

Proof. This follows from combining the results of the previous sections. We choose π as in III.1.3 and $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p))$ as in III.3.1 such that f^p projects to the π^p isotypic piece of $H^*(\mathsf{Sh}, \mathcal{F}_\xi))$. Fix any $h^{\mathbf{G}} \in \mathscr{H}(G(\mathbb{Z}_p) \times G'(\mathbb{Z}_p))$. Note that $\tau \in E_\mu = \mathbf{E}^*_{\mathfrak{p}}$ as discussed in the paragraph before III.2.4.

On the one hand, by III.3.4, we have

$$\mathrm{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = a(\pi_f) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi_p'}(\mathbf{G}_{\mathbb{Q}_p})} \mathrm{tr}(f^{\mathbf{G}}_{\tau, h^{\mathbf{G}}} \mid \pi_p). \tag{200}$$

On the other hand, by II.2.1, we have

$$\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f)), \tag{201}$$

and hence by definition of f^p as well as the argument in III.3.2 using I.6.3,

$$\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \operatorname{tr}(\tau \times h^{\mathbf{G}} \mid \bigoplus_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G})} \pi_p \boxtimes \sigma(\pi_f)). \tag{202}$$

Now, using III.2.6, the above equals

$$a(\pi_f) \operatorname{tr}(\tau \mid (r_{(-\boldsymbol{\mu},0)} \circ \rho|_{W_{\mathbf{E}^*_\mathfrak{p}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi'_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(h^{\mathbf{G}} \mid \pi_p). \tag{203}$$

Finally, by III.2.2, compatibility of the local Langlands correspondence and local base change ([Mok15, Theorem 3.2.1 (a)]), and III.2.4, we have that

$$\begin{aligned} r_{(-\boldsymbol{\mu},0)} \circ \rho|_{W_{\mathbf{E}^*_\mathfrak{p}}} &\cong r_{(-\boldsymbol{\mu},0)} \circ \psi_{BC(\pi)_p}|_{W_{\mathbf{E}^*_\mathfrak{p}}} \\ &\cong r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_\mathfrak{p}}} \\ &= r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_\mathfrak{p}}} \end{aligned} \tag{204}$$

Hence the righthand side of the previous equality becomes

$$a(\pi_f) \operatorname{tr}(\tau | (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_\mathfrak{p}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(h^{\mathbf{G}} \mid \pi_p). \tag{205}$$

Finally, combining the the two equations for $\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi))$ gives that

$$\sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(f^{\mathbf{G}}_{\tau, h^{\mathbf{G}}} \mid \pi_p) \tag{206}$$

is equal to

$$\operatorname{tr}(\tau \mid (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_\mathfrak{p}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(h^{\mathbf{G}} \mid \pi_p). \tag{207}$$

We now need to translate this equation to one for G instead of $\mathbf{G}_{\mathbb{Q}_p}$. Since our choice of $h^{\mathbf{G}}$ was arbitrary, we pick it so that $h^{\mathbf{G}} = h \times h'$ where h' has trace 1 on a single representation in the packet $\Pi_{\psi'_p}(G')$ and trace 0 on the others. We can do this since local A-packets are finite (e.g. see [HG, Proposition 8.5.2]). Since μ is trivial on G', we have that $f_{\tau,h'} = h'$. Indeed, the triviality of μ' implies that the space $\mathscr{D}_\infty(\mathcal{G}', [b'], \mu')$ (where μ' is the projetion to μ) as in [You19] is the trivial $\mathcal{G}'(\mathbb{Z}_p)$-torsor for any $[b']$ as in loc. cit. In particular, this implies that $H^*(\mathscr{D}_\infty(\mathcal{G}', [b'], \mu'), \mathbb{Q}_\ell)$ is nothing more than $C_c^\infty(\mathcal{G}'(\mathbb{Z}_p))$. Since the action of τ is through right multiplication by b' it's clear that the trace of $\tau \times h$ on $\mathscr{D}_\infty(\mathcal{G}', [b'], \mu')$, which is by definition $f_{\tau,h'}(b')$, is just $h'(b')$. Moreover, we have that

$$f^{\mathbf{G}}_{\tau,h\times h'} = f^{G}_{\tau,h} \times f^{G'}_{\tau,h'} = f_{\tau,h} \times h'. \tag{208}$$

as there is a natural splitting of the space

$$\mathscr{D}_\infty(\mathcal{G} \times \mathcal{G}', [(b,b')]\mu) \cong \mathscr{D}_\infty(\mathcal{G}, [b], \mu) \times \mathscr{D}_\infty(\mathcal{G}'[b'], \mu') \tag{209}$$

which is equivariant for the action of $\mathcal{G}(\mathbb{Z}_p) \times \mathcal{G}'(\mathbb{Z}_p)$.

Then, using that $\Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p}) = \Pi_{\psi_p}(G) \times \Pi_{\psi'_p}(G')$, we get

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \mathrm{tr}(\tau \mid (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_{\mathfrak{p}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p). \tag{210}$$

Now we denote by μ', the cocharacter of $G'_{\overline{\mathbb{Q}_p}}$ such that under ι_p, (μ, μ') maps to $\boldsymbol{\mu}$. By construction μ' is trivial and hence $r_{\mu'}$ is the trivial representation. In particular, we get

$$\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi_p) \otimes (r_{-\mu'} \circ \psi'_p)) = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi_p). \tag{211}$$

Making this substitution gives

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi_p)|_{W_{E_\mu}} \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p), \tag{212}$$

as desired. □

Part IV

Appendices

IV.1 Appendix 1: Some lemmas about reductive groups

The goal of this appendix is to collect some loosely related facts about reductive groups, especially with a focus on reductive groups over $\mathbb{R}$.

IV.1.1 Elliptic elements and tori

In this subsection we clarify the relationship between several notions of ellipticity for elements of a reductive group.

So, let us fix a field F of characteristic 0 and let G be a reductive group over F. We begin with the following definition which is unambiguous:

Definition IV.1.1. *A torus T in G containing $Z(G)^\circ$ is said to be* elliptic *if the torus $T/Z(G)^\circ$ is F-anisotropic.*

It is often times the case that a torus T contains not only $Z(G)^\circ$ but $Z(G)$ (e.g. maximal tori). In this case, one might wonder whether one obtains a fundamentally different definition by requiring that $T/Z(G)$ is F-anisotropic. As the following lemma shows, by applying it to the obvious isogeny $T/Z(G)^\circ \to T/Z(G)$, the answer is no. For this reason, we will often times not careful between discussions of the F-anitropicity of $T/Z(G)$ for $T/Z(G)^\circ$ for a torus T containing $Z(G)^\circ$ (again, mostly in the case when T is a maximal torus):

Lemma IV.1.2. *Let T_1 and T_2 be isogenous tori over F. Then, T_1 is F-anisotropic if and only if T_2 is.*

Proof. Let $f : T_1 \to T_2$ be an isogeny. Note then that we get an inclusion $X^*(T_2) \hookrightarrow X^*(T_1)$ with finite cokernel. We and thus an inclusion $X^*(T_2)^\Gamma \hookrightarrow X^*(T_1)^\Gamma$ with finite cokernel. Since $X^*(T_i)^\Gamma$ is free we see that $X^*(T_2)^\Gamma$ is trivial if and only if $X^*(T_1)^\Gamma$ is trivial as desired. $\square$

The definition of what it means for a semisimple element γ in $G(F)$ to be 'elliptic' is a little less clear. Namely, we have the following:

Definition IV.1.3. *A semisimple element γ in $G(F)$ is* elliptic *if $Z(Z_G(\gamma))^\circ$ is an elliptic torus. We will say that such an element γ is* strongly elliptic *if γ is contained in $T(F)$ for some elliptic maximal torus T of G.*

Note that evidently strongly elliptic implies elliptic. Indeed, if T is an elliptic maximal torus such that $\gamma \in T(F)$ then T is a maximal torus in $Z_G(\gamma)$ and thus $\subseteq Z(G)^\circ Z(Z_G(\gamma))^\circ$ is a subtorus of T. Since T is elliptic this implies that $Z(Z_G(\gamma))^\circ$ is elliptic.

Of course, it can't be true in general that elliptic implies strongly elliptic since there are reductive groups which contain no elliptic maximal tori but which contain elliptic elements.

Example IV.1.4. For any perfect field F the maximal tori in $\mathrm{GL}_{n,F}$ are of the form $\prod_{i=1}^{k} \mathsf{Res}_{E_i/F}\, \mathbb{G}_{m,E_i}$ where E_i/F are field extensions and $\sum_{i=1}^{k}[E_i : F] = n$. Moreover, one can check that amongst these the elliptic maximal tori are those of the form $\mathsf{Res}_{E/F}\mathbb{G}_{m,E}$ where $[E : F] = n$. Thus, we see that $\mathrm{GL}_{n,F}$ has an elliptic maximal torus if and only if F admits an extension of degree n.

In particular, we see that $\mathrm{GL}_{n,\mathbb{R}}$ admits an elliptic maximal torus if and only if $n = 2$. That said, $\mathrm{GL}_{n,\mathbb{R}}$ has elliptic elements for all $n \geqslant 1$. Indeed, for any group G the identity element $G(F)$ is elliptic.

That said, in most of the cases of interest to us the definitions coincide. For instance, we have the following observation:

Proposition IV.1.5. *Let F be a p-adic local field. Then, a semisimple element γ in $G(F)$ is elliptic if and only if it's strongly elliptic.*

Lemma IV.1.6. *Let F be a p-adic local field and let H be a reductive group over F. Then, H contains an elliptic maximal torus.*

Proof. By [PS92, Theorem 6.21] we know that $H/Z(H)$ contains a maximal anisotropic torus T. Evidently the preimage of T under the projection map $H \to H/Z(H)$ produces the desired elliptic maximal torus. $\square$

Proof. (Proposition IV.1.5) As we've already observed, it suffices to show that if $\gamma \in G(F)$ is elliptic, then it's strongly elliptic. That said, note that $H := I_\gamma$ contains an elliptic maximal torus T which is evidently a maximal torus of G since H contains a maximal torus of G and thus has the same rank as G. By definition, this implies that $T/Z(H)$ is F-anistropic. That said note that by our assumption the split rank of $Z(H)$ and the split rank of $Z(G)$ coincides. Thus, $T/Z(H)$ having split rank 0 implies that $T/Z(G)$ has split rank 0. Since γ is contained in $T(F)$ the claim follows. $\square$

We would like to extend this result to all characteristic 0 local fields and so, in particular, extend this result to $\mathbb{R}$ (note that the only elliptic torus in a group G over $\mathbb{C}$ is $Z(G)^\circ$). But, as we observed in Example IV.1.4 such a result fails for trivial reasons over $\mathbb{R}$ for general groups. That said, one can ask whether the notion of elliptic and strongly elliptic do agree for semisimple elements in $G(\mathbb{R})$ where G is a reductive group over $\mathbb{R}$ that does contain an elliptic maximal torus. The answer is yes.

To see this, we begin with the following well-known result:

Lemma IV.1.7. *Let G be a reductive group over $\mathbb{R}$. Then, for every compact subgroup K contained in $G(\mathbb{R})$ there exists an $\mathbb{R}$-anisotropic group H and a closed embedding $H \hookrightarrow G$ such that $H(\mathbb{R}) = K$.*

Proof. This is [Ser93, §5 Proposition 2]. $\square$

One consequence of this is the following:

Lemma IV.1.8. *Let G be a reductive group over $\mathbb{R}$. Then, all maximal anisotropic tori in G are conjugate. Moreover, all maximal elliptic tori in G are conjugate.*

Proof. Let us begin by showing that the former statement implies the latter. Namely, let T_1 and T_2 be two maximal elliptic tori in G. Note then that by standard theory we have a decomposition $T_i = T_i^s T_i^a$ where T_i^s is the maximal split subtorus of T_i and T_i^a is the maximal anisotropic subtorus. Moreover, we have that $T_i^s \cap T_i^a$ is finite. Note that by our ellipticity assumptions we have that $T_i^s = (Z(G)^\circ)^s$ for $i = 1, 2$.

Let us note that T_i^a are maximal aniostropic tori in G, as we now show. By symmetry we need only consider the case when $i = 1$. Now, suppose that T is an anisotropic torus of G strictly containing T_1^a. Then, evidently $TZ(G)^\circ$ is an elliptic torus of G strictly containing T_1 which contradicts assumptions.

So, assuming that all anisotropic tori in G are conjugate there exists some $g \in G(\mathbb{R})$ such that $gT_1^a g^{-1} = T_2^a$. Note then evidently that since conjugation by g fixes $Z(G)$ pointwise that

$$gT_1g^{-1} = g(T_1^a Z(G)^\circ)g^{-1} = T_2^a Z(G)^\circ = T_2 \tag{213}$$

as desired.

Suppose now that T_1 and T_2 are maximal anisotropic tori in G. Note then that $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are compact subgroups of G and thus contained in maximal compact subgroups K_1 and K_2 of $G(\mathbb{R})$. Now, it is well-known (e.g. see [Con14, Theorem D.2.8]) that K_1 and K_2 are conjugate by an element of $G(\mathbb{R})$. Thus without loss of generality we may assume the equality $K := K_1 = K_2$. Moreover, by Lemma IV.1.7 we know that $K = H(\mathbb{R})$ for H some $\mathbb{R}$-anisotropic subgroup of G.

We claim that both $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are maximal tori in K in the sense of the theory of compact Lie groups (i.e. that they are maximal connected compact abelian subgroups). Indeed, suppose not. Then there exists a connected compact abelian subgroup $S \subseteq K = H(\mathbb{R})$ properly containing $T_1(\mathbb{R})$. But, by [Con14, Theorem D.2.4] this implies that there exists some connected $\mathbb{R}$-anisotropic group $S^{\mathrm{alg}} \subseteq H$ such that $S^{\mathrm{alg}}(\mathbb{R}) = S$. Note then that by the Zariski denseness of $\mathbb{R}$-points (e.g. see [Mil17, Theorem 17.9.3]) we have that S^{alg} properly contains T_1. But, since S is dense in S^{alg} we see that S^{alg} is necessarily abelian. Thus, S^{alg} is an anisotropic torus in H properly containing T_1. This contradicts that T_1 is a maximal anisotropic torus in G. By symmetry the claim also applies for T_2.

Thus, since $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are maximal tori in K in the sense of the theory of compact Lie groups we know from the theory of such groups that $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are conjugate by an element of K. Then, again by density of $T_1(\mathbb{R})$ in T_1, we deduce that T_1 is conjugate to T_2. More rigorously let $g \in K = H(\mathbb{R})$ conjugate $T_1(\mathbb{R})$ to $T_2(\mathbb{R})$. Note then that conjugation map by g sends $T_1(\mathbb{R})$ into $T_2 \subseteq G$ from which density of $T_1(\mathbb{R})$ in T_1 implies that conjugation by g takes T_1 into T_2. This implies that $\dim T_1 \leqslant \dim T_2$. By symmetry we deduce that $\dim T_2 \leqslant \dim T_1$. Then, since $gT_1g^{-1} \subseteq T_2$ and gT_1g^{-1} and T_2 are both connected and smooth we deduce that $gT_1g^{-1} = T_2$ as desired. $\square$

Two important corollaries of this result are the following:

Corollary IV.1.9. *Let G Be a reductive group over $\mathbb{R}$ and suppose that G has an elliptic maximal torus. Then, every maximal elliptic torus in G is an elliptic maximal torus.*

Corollary IV.1.10. *Let G be a reductive group over $\mathbb{R}$ and suppose that G has an elliptic maximal torus T_0. Then, every elliptic element γ in $G(\mathbb{R})$ is strongly elliptic.*

Proof. Note that, by definition, γ is contained in an elliptic torus T_1 of G (namely $T_1 = Z(Z_G(\gamma))^\circ$). Note then that T_1 is contained in some maximal elliptic torus T of G. But, by the previous corollary T is a maximal torus in G. The conclusion follows. $\square$

We finally record the following well-known results concerning the existence of elliptic maximal tori in groups over $\mathbb{R}$. Namely, while it is classical that every reductive group G over $\mathbb{R}$ admits a unique anisotropic form. That said, the existence of an anisotropic modulo center inner form is not guaranteed and is related to the existence of an elliptic maximal torus. Namely:

Lemma IV.1.11. *Let G be a connected reductive group over $\mathbb{R}$. Then, G admits an elliptic maximal torus if and only if G admits an anisotropic modulo center inner form.*

IV.1.2 Local-to-global construction of elliptic maximal tori

In this subsection we would like to verify that if G is a reductive group over a number field F we can construct maximal tori in G which become elliptic over some some finite set of places S of F as long as there are no tautological obstructions (i.e. that G has no elliptic maximal tori at one of the places in S). More rigorously:

Proposition IV.1.12. *Let F be a number field and let G be a connected reductive group over F. Suppose that S is a finite set of places of F such that for all $v \in S$ the group G_{F_v} contains an elliptic maximal torus. Then, there exists a maximal torus T in G such that T_{F_v} is an elliptic maximal torus in G_{F_v} for all $v \in S$.*

To prove this it will be helpful to set up some notation and recall some classical results concerning the moduli of maximal tori in G. For now, let F be any field of characteristic 0 and let G be a connected reductive group over F. To begin, let us define X to be the functor associating to an F-algebra R the set $X(R)$ of maximal tori in G_R (e.g. in the sense of [Con14, Definition 3.2.1]). Then, we have the following result:

Lemma IV.1.13. *The functor X is represented by a smooth, irreducible, and quasi-affine F-scheme (also denoted X). Moreover, for any maximal torus T_0 in G there is a canonical isomorphism $G/N_G(T_0) \to X$.*

Proof. See [Con14, Theorem 3.2.6] for the first statement minus the smoothness and irreducibility and the second statement. Note that the conditions that the maximal tori in $G_{\overline{F}}$ are self-centralizing follows immediately from the reductive hypotheses on G. The smoothness and irreducibility of X then follow *a fortiori* from the second statement given the smoothness and irreducibility of G. □

We shall need the following structural result of Chevalley concerning X:

Theorem IV.1.14 (Chevalley)**.** *The scheme X is F-rational.*

Now, for any field F' containing F let us denote by $X^e(F')$ the subset of $X'(F)$ consisting of F'-elliptic maximal tori in $G_{F'}$. Be careful that, despite the notation, $X^e(F')$ is evidently not functorial in F'.

We then have the following observation:

Lemma IV.1.15. *Suppose that F is a characteristic 0 local field. Then, $X^e(F)$ is an open (possibly empty) subset of $X(F)$ where the latter is endowed with the usual topology F-topology.*

Proof. Let us denote by $\mathbb{T}$ the universal maximal torus over X. For a point $x \in X(F)$ we denote by $\mathbb{T}_x$ the corresponding torus of G since split rank is an isogeny invariant (e.g. see Lemma IV.1.2). It then suffices to show that the isogeny class of $\mathbb{T}_x$ is locally constant in x. To do this we proceed as follows. Let us note that X is rational and smooth, so that $\mathbb{T}$ gives rise (by [Con14, Corollary B.3.6]) to a continuous representation $\pi_1(X, \overline{x_0}) \to \mathrm{GL}_n(\mathbb{Z})$ (where n is the rank of $\mathbb{T}$).

Note that this representation must factor through a finite quotient Q of $\pi_1(X, \overline{x_0})$. Note that for $x \in X(F)$ the torus $\mathbb{T}_x$ clearly corresponds to the composition $\Gamma_F \to \pi_1(X, \overline{x_0}) \to \mathrm{GL}_n(\mathbb{Z})$ which we denote ρ_x. Note, in particular that for any $x \in X(F)$ we have that ρ_x has image bounded by $|Q|$ and so Γ_F factors through a quotient of size $|Q|$. Since F has only finitely many extensions of size $|Q|$ we see that there must be some finite extension F'/F such that ρ_x factors through $\mathrm{Gal}(F'/F)$ for all $x \in X(F)$.

Let us denote, for each $x \in X(F)$, the composition of ρ_x with the embedding $\mathrm{GL}_n(\mathbb{Z}) \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$ by $\rho_x^{\mathbb{Q}}$. Then, by the Brauer-Nesbitt theorem we know that $\rho_x^{\mathbb{Q}} \cong \rho_{x'}^{\mathbb{Q}}$ if and only if $\chi_{\rho_x(g)} = \chi_{\rho_{x'}(g)}$ for all $g \in \mathrm{Gal}(F'/F)$ where we have used χ_T to denote the characteristic polynomial for T. But, since the coefficients of ρ_x are roots of unity, we know that $\chi_{\rho_x(g)} = \chi_{\rho_x(g')}$ if and only if they agree modulo N for N sufficiently large. In other words, we see that if $\mathbb{T}_x[N] \cong \mathbb{T}_{x'}[N]$ then $\mathbb{T}_x$ and $\mathbb{T}_{x'}$ are isogenous.

Let us then pick a point $x \in X(F)$ and consider the finite étale cover $\underline{\mathrm{Isom}}(\mathbb{T}[n], \mathbb{T}_{x_0}[N])$ of X. Note then that since the point $x_0 \in X(F)$ has a lift to a point of $\underline{\mathrm{Isom}}(\mathbb{T}[n], \mathbb{T}_{x_0}[N])(F)$ then by standard theory (e.g. see [Poo17, Theorem 3.5.73.(i)]) there exists a neighboorhod U of x_0 in $X(F)$ such that $\underline{\mathrm{Isom}}(\mathbb{T}[N], \mathbb{T}_{x_0}[N])(F) \to X(F)$ admits a section. By the above, this implies that $\mathbb{T}_x$ is isogenous to $\mathbb{T}_{x_0}$ for all $x \in U$, and so the conclusion follows. □

Using the above results we can now prove Proposition IV.1.12:

Proof. (Proposition IV.1.12) Let us denote by F_S the usual F-algebra $\prod_{v \in S} F_v$. Note then that we have a natural diagonal embedding $X(F) \to X(F_S)$. Moreover, since X is F-rational, smooth, and irreducible we know that the image of $X(F)$ in $X(F_S)$ is dense (e.g. see [PS92, Proposition 7.3]). Now, by assumption we have that $X^e(F_v)$ is non-empty for all $v \in S$ and thus combining this with Lemma IV.1.15 we see that $\prod_{v \in S} X^e(F_v)$ is a non-empty open subset of $X(F_S)$. Since $X(F)$ is a dense subset of $X(F_S)$ we thus deduce that $X(F)$ and $\prod_{v \in S} X(F_v)$ must have a point in common. The conclusion follows. □

IV.1.3 Stable conjugacy for strongly regular elements over $\mathbb{R}$

The goal of this subsection is to clarify the nature of stable conjugacy for strongly regular elements in $G(\mathbb{R})$ where G is a reductive group over $\mathbb{R}$.

Before we begin, let us fix some notation that will be used below (as well as the main body of the paper).

Definition IV.1.16. *Let T be a maximal torus in G. For any Levi subgroup M of G containing T we denote by $W(M,T)$ the Weyl group scheme $N_M(T)/T$. We will denote by $W_{\mathbb{C}}(M,T)$ the group*

$$W_{\mathbb{C}}(M,T) := N_G(T)(\mathbb{C})/T(\mathbb{C}) = W(M,T)(\mathbb{C}) \tag{214}$$

We denote by $W_{\mathbb{R}}(M,T)$ the group

$$W_{\mathbb{R}}(M,T) := N_G(T)(\mathbb{R})/T(\mathbb{R}) \subseteq W(M,T)(\mathbb{R}) \tag{215}$$

where this last containment can be strict in general. When $M = G$ we use the shortenings $W_{\mathbb{C}}$ and $W_{\mathbb{R}}$ of the above notation.

Remark IV.1.17. For the sake of notational comparison, let us note that if T is an elliptic maximal torus then $W_\mathbb{R}$ is often written (for example in Harish-Chandra's parametrization of discrete series) as W_c and called the *compact Weyl group*. The reason is that in this case $W_\mathbb{R}$ is equal to $W(K, T(\mathbb{R}))$ for any maximal compact subgroups of $G(\mathbb{R})$ containing $T(\mathbb{R})$. The reason of course, is that $N_G(T)(\mathbb{R})$, containing $T(\mathbb{R})$ as a finite index subgroup, is itself compact and so contained in a maximal compact subgroup of $G(\mathbb{R})$.

We also recall the following well-known definitions:

Definition IV.1.18. *Let G be a reductive group over a field F. A semsimple element γ in $G(F)$ is* regular *if I_γ is a (necessarily maximal) torus of G. We say that γ is* strongly regular *if $Z_G(\gamma)$ is a (necessarily maximal) torus of G.*

Recall that if G^{der} is simply connected then these two notions coincide. Indeed, in the case by the following well-known result of Steinberg:

Theorem IV.1.19 (Steinberg)**.** *Let G be a reductive group over a field F and assume that G^{der} is simply connected. Then, for any semisimple $\gamma \in G(F)$ we have that $Z_G(\gamma)$ is connected.*

Proof. To show that $Z_G(\gamma)$ is connected it suffices to show that $Z_G(\gamma)_{\overline{F}}$ is connected, and so it suffices to assume that F is algebraically closed. Note that we have a short exact sequence of groups

$$0 \to G^{\mathrm{der}} \to G \to G^{\mathrm{ab}} \to 0 \tag{216}$$

Note that since G is reductive we have that $G = G^{\mathrm{der}} Z(G)$ and so $Z(G)$ surjects onto G^{ab}. Since $Z_G(\gamma) \supseteq Z(G)$ we deduce that $Z_G(\gamma)$ surjects onto G^{ab}. Thus, the sequence (216) gives rise to the sequence

$$0 \to G^{\mathrm{der}} \cap Z_G(\gamma) \to Z_G(\gamma) \to G^{\mathrm{ab}} \to 0 \tag{217}$$

Thus, since G^{ab} is connected since G is, it suffices to show that $G^{\mathrm{der}} \cap Z_G(\gamma)$ is connected. Note that since $G = G^{\mathrm{der}} Z_G(\gamma)$ that there exists some $z \in Z(G)(F)$ such that $\gamma z \in G^{\mathrm{der}}(F)$. Clearly $Z_G(\gamma) = Z_G(\gamma z)$ and so it suffices to assume that $\gamma \in G^{\mathrm{der}}(F)$. Note then that $G^{\mathrm{der}} \cap Z_G(\gamma) = Z_{G^{\mathrm{der}}}(\gamma)$. Thus, it finally suffices to assume that $G = G^{\mathrm{der}}$. In this setting one can find a proof in [Ste06, §5] or [Hum11, §2.11] $\square$

It will also be helpful to record the following basic observation:

Theorem IV.1.20 (Steinberg)**.** *Let G be a reductive group over a field F. Then, the set U of regular elements of G is an open subset of F. In particular, $U(F)$ is dense in G.*

Proof. The fact that U is open follows from [Ste65, 1.3]. Note then that since G is unirational (e.g. see [Mil17, Theroem 17.93]) the same is true for U. Thus, $U(F)$ is Zariski dense in U. But, since U is open in G and G is irreducible (e.g. by [Mil17, Summary 1.36]) we know that U is dense in G so that $U(F)$ is dense in G as desired. $\square$

We now state our target proposition:

Proposition IV.1.21. *Let G be a reductive group over $\mathbb{R}$ and let T be a maximal torus in $\mathbb{R}$. Let S be a maximal split subtorus of T and set $M := Z_G(S)$. Let $\gamma \in T(\mathbb{R})$ be strongly regular. Then:*

$$\{\gamma\}_s = \bigcup_{w \in W_{\mathbb{C}}(M,T)} \{w\gamma w^{-1}\} = \bigcup_{w \in W_{\mathbb{C}}(M,T)/W_{\mathbb{R}}(M,T)} \{w\gamma w^{-1}\} \tag{218}$$

An immediate corollary, the case of most interest to us, is the following:

Corollary IV.1.22. *Let G be a reductive group over $\mathbb{R}$ and suppose that T is a maximal elliptic torus then*

$$\{\gamma\}_s = \bigcup_{w \in W_{\mathbb{C}}} \{w\gamma w^{-1}\} = \bigsqcup_{w \in W_{\mathbb{C}}/W_{\mathbb{R}}} \{w\gamma w^{-1}\} \tag{219}$$

Proof. This follows immediately from the proposition since one can take S to be a maximal split subtorus of $Z(G)$ so that $M = G$. $\square$

Example IV.1.23. Let $G = \mathrm{SL}_{2,\mathbb{R}}$. Then, the classic example of two non-conjugate but stably conjugate elements of $\mathrm{SL}_2(\mathbb{R})$ is $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note though that $\gamma \in T(\mathbb{R})$ where T is the elliptic maximal torus

$$T = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} \subseteq \mathrm{SL}_{2,\mathbb{R}} \tag{220}$$

Moreover, note that $|W_{\mathbb{C}}| = 2$ with the non-trivial class represented by $w := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Moreover, it's not hard to check that

$$\mathrm{Int}(w) : T \to T \tag{221}$$

is given by

$$\mathrm{Int}(w) : \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{222}$$

Thus, the above corollary shows that

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\}_s = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\} \tag{223}$$

and thus

$$\{\gamma\}_s = \{\gamma\} \cup \{\gamma'\} \tag{224}$$

explaining the above example.

Let us begin by clarifying how $\{w\gamma w^{-1}\}$ makes sense for $w \in N_M(T)(\mathbb{C})$ as an element of $\{G\}$. This is settled by the following:

Lemma IV.1.24 ([She79b, Theorem 2.1])**.** *Let notation be as in the beginning previous proposition. Then, the group*

$$\{g \in G(\mathbb{C}) : \mathrm{Int}(g) : T_{\mathbb{C}} \to G_{\mathbb{C}} \text{ is defined over } \mathbb{R}\} \tag{225}$$

is equal to the group $G(\mathbb{R})N_M(T)(\mathbb{C})$.

In particular, for any $\gamma \in T(\mathbb{R})$ and $g \in N_M(T)(\mathbb{C})$ we have that the map $\mathrm{Int}(g) : T_{\mathbb{C}} \to T_{\mathbb{C}}$ is defined over $\mathbb{R}$, and thus $g\gamma g^{-1}$ is an element of $T(\mathbb{R})$. Thus, $\{g\gamma g^{-1}\}$ is a well-defined element of $\{G\}$.

Remark IV.1.25. Note that, a priori, the conjugacy class $\{g\gamma g^{-1}\}$ may depend on the choice of γ in $\{\gamma\}$. Thus, the notation $w \cdot \{\gamma\}$ doesn't a priori make sense for $w \in W_C(M,T)$. In fact, the well-definedness of $w \cdot \{\gamma\}$ (the independence of choice representative in $\{\gamma\}$ in $T(\mathbb{R})$) is equivalent to the normality of $W_{\mathbb{R}}(M,T)$ in $W_{\mathbb{C}}(M,T)$ which needn't necessarily hold. That said, the right-hand side of (218) doesn't depend on a choice of γ.

To begin to prove Proposition IV.1.21 we begin with the following observation:

Lemma IV.1.26. *Suppose that $\gamma \in T(\mathbb{R})$ is strongly regular. Suppose that $\gamma' \in G(\mathbb{R})$ is stably conjugate to γ. Then, γ' is strongly regular and the tori $T' := Z_G(\gamma')$ and T are stably conjugate (i.e there is a $g \in G(\mathbb{C})$ such that $\mathrm{Int}(g) : T_{\mathbb{C}} \to T'_{\mathbb{C}}$ and the map is defined over $\mathbb{R}$).*

Proof. The fact that γ' is strongly regular is clear since $Z_G(\gamma')$ and $Z_G(\gamma)$ are forms of each other, and thus $Z_G(\gamma')$ is a torus. Now, by assumption, there is $g \in G(\mathbb{C})$ such that $g\gamma g^{-1} = \gamma'$. In particular, for $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$,

$$\begin{aligned} \sigma(g)\gamma\sigma(g)^{-1} &= \sigma(g\gamma g^{-1}) \\ &= \sigma(\gamma') \\ &= \gamma' \\ &= g\gamma g^{-1} \end{aligned} \tag{226}$$

Hence, $\sigma(g)^{-1}g = t_1 \in T(\mathbb{C})$.

Now, we need to show $\mathrm{Int}(g) : T \to T'$ is defined over $\mathbb{R}$. In particular, we need to show that $\sigma \circ \mathrm{Int}(g) \circ \sigma^{-1} = \mathrm{Int}(g)$. But we have for all $t \in T(\mathbb{C})$,

$$\begin{aligned}(\sigma \circ \mathrm{Int}(g) \circ \sigma^{-1})(t) &= \sigma(g\sigma^{-1}(t)g^{-1}) \\ &= \sigma(g)t\sigma(g)^{-1} \\ &= gt_1^{-1}tt_1g^{-1} \\ &= gtg^{-1} \\ &= \mathrm{Int}(g)(t)\end{aligned} \tag{227}$$

from where the result follows since T and T' are separated. □

The last preliminary result we need is the following:

Theorem IV.1.27 ([She79a, Cor 2.3])**.** *Let G be a reductive group over $\mathbb{R}$ and let T and T' be maximal tori in G. Then, if T and T' are stably conjugate, then they are conjugate.*

We now prove the main proposition as follows:

Proof. (Proposition IV.1.21) Evidently

$$\{\gamma\}_s \supseteq \bigcup_{w \in W_C(M,T)} \{w\gamma w^{-1}\} \tag{228}$$

Conversely, suppose that $\gamma' \in G(\mathbb{R})$ is stably conjugate to γ. Since γ is strongly regular we know from Lemma IV.1.26 that T and $T' := Z_G(\gamma')$ are stably conjugate. Thus, by Lemma IV.1.27 we know that T and T' are conjugate. Thus, we may assume without loss of generality (without changing the conjugacy class) that $\gamma' \in T(\mathbb{R})$. Let $g \in G(\mathbb{C})$ be such that $g\gamma g^{-1} = \gamma'$. Since γ is strongly regular this implies, by Lemma IV.1.26, that $\mathrm{Int}(g)$ maps $T_{\mathbb{C}} \to T_{\mathbb{C}}$ and, in fact, is defined over $\mathbb{R}$. By Lemma IV.1.24 this implies that $g \in G(\mathbb{R})N_M(T)(\mathbb{C})$. But, since conjugation by $G(\mathbb{R})$ evidently doesn't effect conjugacy classes, we may assume that $g \in N_M(T)(\mathbb{C})$. The first part of (218) follows.

Suppose now that $w_1\gamma w_1^{-1}$ is conjugate to $w_2\gamma w_2^{-1}$. Then, there exists some $g \in G(\mathbb{R})$ such that

$$w_2\gamma w_2^{-1} = gw_1\gamma w_1^{-1}g^{-1} \tag{229}$$

so that $g \in N_G(T)(\mathbb{R})$ and $w_2^{-1}gw_1$ fixes γ. Since γ is strongly regular this implies that $w_2^{-1}gw_1 \in T(\mathbb{R})$ which means that $w_2^{-1}gw_1$ is the trivial element of $W_{\mathbb{C}}$. This says that $w_2 = gw_1$ as elements of $W_{\mathbb{C}}$. Since $g \in N_G(T)(\mathbb{R})$ we see that $g \in W_{\mathbb{R}}$ and the second equality of (218) follows. □

IV.1.4 Reflex fields and a construction of Kottwitz

In this appendix we record, for the ease of the reader, the following extension of a classicl construction of Kottwitz (see [Kot84a, Lemma 2.1.2]) to the setting of not necessarily quasi-split groups.

Let us fix a field F and G a reductive group over F. Let $\boldsymbol{\mu}$ be a conjugacy class of cocharacters over $\overline{F}$. Recall that Γ_F acts on the set of conjugacy class of cocharacters of $G_{\overline{F}}$ and we define the *reflex field* of $\boldsymbol{\mu}$, denoted by $E(\boldsymbol{\mu})$ (or just E when $\boldsymbol{\mu}$ is clear from context), to be the fixed field.

Let G^* denote the quasi-split inner form of G over F. Choose an inner twisting $f : \mathbf{G}_{\overline{F}} \to \mathbf{G}^*_{\overline{F}}$ and let us specify that $\sigma \mapsto g_\sigma$ is the $G^{\mathrm{ad}}(\overline{F})$-valued cocycle such that for all $\sigma \in \Gamma_F$ we have that

$$f \circ \sigma_{G_{\overline{F}}} \circ f^{-1} \circ \sigma^{-1}_{G^*_{\overline{F}}} = \mathrm{Inn}(g_\sigma)$$

We then have the following observation:

Lemma IV.1.28. *The reflex field of the $G^*(\overline{F})$-conjugacy class of cocharacters*

$$f(\boldsymbol{\mu}) := \{f \circ \mu : \mu \in \boldsymbol{\mu}\}$$

is $E(\boldsymbol{\mu})$.

Proof. To see this it suffices to show that for any σ in Γ_F we have that $\sigma \cdot (f \circ \mu)$ is conjugate to $f \circ \mu$ since, by symmetry, the reverse direction will also follow. To see this we merely note that for any $\sigma \in \Gamma$ we have that

$$\begin{aligned}\sigma \cdot (f \circ \mu) &= \sigma_{G^*_{\overline{F}}} \circ f \circ \mu \circ \sigma^{-1}_{\mathbb{G}_{m,\overline{F}}} \\ &= \mathrm{Inn}(g_\sigma^{-1}) \circ f \circ \sigma_{G_{\overline{F}}} \circ \mu \circ \sigma^{-1}_{\mathbb{G}_{m,\overline{\mathbb{Q}}}} \\ &= \mathrm{Inn}(g_\sigma^{-1}) \circ f \circ \mathrm{Inn}(h_\sigma) \circ \mu) \\ &= \mathrm{Inn}(g_\sigma^{-1} f(h_\sigma)) \circ f \circ \mu\end{aligned}$$

where we have used the fact that $\boldsymbol{\mu}$ is Γ_F-stable to obtain the element h_σ. $\square$

It's also clear that if we choose another inner twisting (G^*, f') of G that $f'(\boldsymbol{\mu}) = f(\boldsymbol{\mu})$ since for all μ in $\boldsymbol{\mu}$ we have that $f \circ \mu$ is conjugate to $f' \circ \mu$ by definition. Thus, we see that this conjugacy class of cocharacters of $G^*_{\overline{F}}$ depends only on G^* and not on the inner twist (G, f). Thus, we denote this conjugacy class $\boldsymbol{\mu}^*$. By the above we have that $E(\boldsymbol{\mu}) = E(\boldsymbol{\mu}^*)$. Also note that for any $\boldsymbol{\mu}$ we have that $(-\boldsymbol{\mu})^* = -\boldsymbol{\mu}^*$.

Let us now choose a rational Borel-torus pair (B, T) of G^* over F. To $\boldsymbol{\mu}^*$ we associate a $\overline{\mathbb{Q}_\ell}$-representation $r_{\boldsymbol{\mu}}$ of $\widehat{\mathbf{G}^*} \rtimes W_{E(\boldsymbol{\mu}^*)}$ where $W_{E(\boldsymbol{\mu})^*}$ acts on $\widehat{\mathbf{G}^*}$ via the pair (B, T). To do this note that since $\mathbf{G}^*$ is quasi-split we have that $\boldsymbol{\mu}$ is actually defined over $E(\boldsymbol{\mu})$ (see [Kot84a, Lemma 1.1.3]). Let μ be the unique B-dominant representative of $\boldsymbol{\mu}^*$ defined over $E(\boldsymbol{\mu}^*)$. Let

$V(\mu)$ be the irreducible $\overline{\mathbb{Q}_\ell}$-representation with highest weight μ and then define

$$r_{\boldsymbol{\mu}^*} : \widehat{\mathbf{G}^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \to \mathrm{GL}(V(\mu))$$

to be such that its restriction to $\widehat{\mathbf{G}^*}$ is the usual action and such that the action of $W_{E(\boldsymbol{\mu}^*)}$ on the weight space $V_\mu \subseteq V(\mu)$ is trivial. The existence of such a representation is precisely [Kot84a, Lemma 2.1.2].

Note though that there is an isomorphism

$$\widehat{G^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \cong \widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \tag{230}$$

unique up to inner automorphism. Thus, associated to $r_{\boldsymbol{\mu}^*}$ is a representation

$$\widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \xrightarrow{\approx} \widehat{G^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \to \mathrm{GL}(V(\mu)) \tag{231}$$

unique up to isomorphism which we denote $r_{\boldsymbol{\mu}}$. Of course, up to isomorphism, this representation doesn't depend on the choice of (B, T) and, in particular, depends only on $\boldsymbol{\mu}$ not the choice of an element $\mu \in \boldsymbol{\mu}$. Thus, we will often times write $r_{\boldsymbol{\mu}}$ as a representation $\widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \to \mathrm{GL}(V(\boldsymbol{\mu}))$.

We now record some results in the case of F being a global field. To begin we note thatfor any place v of F and any choice of embedding $\overline{F} \hookrightarrow \overline{F_v}$ one gets an induced conjugacy class $\boldsymbol{\mu}_v$ of cocharacters of $G_{\overline{F_v}}$. The following claim is then simple:

Lemma IV.1.29. *There is an equality of fields $E(\boldsymbol{\mu})_w = E(\boldsymbol{\mu}_v)$.*

In particular, we see the following:

Corollary IV.1.30. *Let v be an element of $S^{\mathrm{ur}}(\mathbf{G})$. Then, $E(\boldsymbol{\mu})_w/F_v$ is unramified.*

Proof. Note that by Lemma IV.1.29 it suffices to show that $E(\boldsymbol{\mu}_v)/F_v$ is unramified. But, since $\mathbf{G}_v$ splits over F_v^{ur} we evidently have an inclusion $E(\boldsymbol{\mu}_v) \subseteq F_v^{\mathrm{ur}}$ from where the claim follows. ☐

The following lemma is equally as simple as Lemma IV.1.29:

Lemma IV.1.31. *There is an equality (up to isomorphism) of representations*

$$r_{\boldsymbol{\mu}} \mid_{\widehat{G} \rtimes W_{E(\boldsymbol{\mu})_w}} = r_{\boldsymbol{\mu}_v} \tag{232}$$

IV.1.5 The Kottwitz group

We record in this section, for the convenience of the reader, the basic definitions and properties we would like to use concerning the *Kottwitz group* associated to a local or global field F.

To make sense of the definition of this group, it is useful to first recall the following basic lemma:

Lemma IV.1.32. *Let F be a field of characteristic 0 and let G be a connected reductive group over F. Let H be any connected reductive subgroup of G of the same rank. The choice of a maximal torus T in H induces a natural Γ_F-equivariant inclusion $Z(\widehat{G}) \subseteq Z(\widehat{H})$, and this embedding is, in fact, independent of T.*

Remark IV.1.33. See [Bor79, §2] for a recollection of dual groups and their associated Galois actions.

Proof. (Lemma IV.1.32) Let us first consider the case when H is a maximal torus defined over F, in which case we will take T to be equal to H. Then, essentially by definition of the dual group, there exists an embedding $\widehat{H} \hookrightarrow \widehat{G}$ of complex algebraic groups identifying the image of $\widehat{H}$ with a maximal torus of $\widehat{G}$. In particular, we see that the image of $\widehat{H}$ contains $Z(\widehat{G})$. Let us denote by Z' the preimage of $Z(\widehat{G})$ in $\widehat{H}$. We then claim that the isomorphism of complex algebraic groups $Z' \to Z(\widehat{G})$ is actually Γ-equivariant.

To see this, note that induced map of root datum from the morphism $\widehat{H} \hookrightarrow \widehat{G}$ can be identified with the natural inclusion

$$(X_*(H), 0, X^*(H), 0) \hookrightarrow (X_*(H), \Phi^\vee(G), X^*(H), \Phi(G)) \tag{233}$$

which is patently Γ-equivariant. Thus, we see that for all $\gamma \in \Gamma$ the action of γ on $\widehat{H}$ and the action of γ on the image of $\widehat{H}$ in $\widehat{G}$ differ by inner automorphisms of G. In particular, it follows that the map $Z' \to \widehat{G}$ is Γ-equivariant, and thus is the map $Z' \to Z(\widehat{G})$, as claimed.

The desired Γ-equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H}) = \widehat{H}$ can thus be taken to be the inverse of the induced Γ-equivariant isomorphism $Z' \xrightarrow{\approx} Z(\widehat{G})$ discussed above.

Suppose now that H is an arbitrary reductive subgroup of G of the same rank. Let us fix a maximal torus T of H. From the initial case when H was assumed to be a torus, we see that we obtain separate Γ-equivariant embeddings $Z(\widehat{G}) \hookrightarrow \widehat{T}$ and $Z(\widehat{H}) \hookrightarrow \widehat{T}$. But, since $Z(\widehat{G})$ is clearly contained in $Z(\widehat{H})$ as complex algebraic subgroups of $\widehat{T}$ we thus obtain a Γ-equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ as desired.

Finally, observe that changing the maximal torus T to T' doesn't affect the embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ since $\widehat{T}$ and $\widehat{T'}$ are conjugate in $\widehat{H}$ and this conjugation doesn't alter the embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$. ☐

Suppose now that F is a number field and $\mathbf{G}$ is a reductive group over F. Assume further that $\mathbf{H}$ is a reductive subgroup of G of the same rank. Clearly then for all places v of F we have that $\mathbf{H}_v$ is a reductive subgroup of $\mathbf{G}_v$ of the same rank. Thus, from Lemma IV.1.32 we obtain a Γ_F-equivariant inclusion $Z(\widehat{\mathbf{G}}) \hookrightarrow Z(\widehat{\mathbf{H}})$ and Γ_{F_v}-equivariant inclusions $Z(\widehat{\mathbf{G}_v}) \hookrightarrow Z(\widehat{\mathbf{H}_v})$ for all places v of F. Given our particular embeddings of $\overline{F} \hookrightarrow \overline{F_v}$ we obtain a diagram

$$
\begin{tikzcd}
Z(\widehat{\mathbf{G}_v}) \arrow[r] \arrow[d, "\wr"'] & Z(\widehat{\mathbf{H}_v}) \arrow[d, "\wr"] \\
Z(\widehat{\mathbf{G}}) \arrow[r] & Z(\widehat{\mathbf{H}})
\end{tikzcd}
\tag{234}
$$

where the vertical maps are isomorphisms of complex Lie groups equivariant for the Γ_v action where $Z(\widehat{\mathbf{G}})$ is endowed with the Γ_v action inherited from the inclusion $\Gamma_v \subseteq \Gamma$ induced by our choice of embedding $\overline{F} \hookrightarrow \overline{F_v}$.

From the maps $Z(\widehat{\mathbf{G}}) \to Z(\widehat{\mathbf{H}})$ of Γ-modules obtain a short exact sequence of Γ-modules

$$
0 \to Z(\widehat{\mathbf{G}}) \to Z(\widehat{\mathbf{H}}) \to Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}) \to 0 \tag{235}
$$

Moreover, for each place v of F we obtain from the map $Z(\widehat{\mathbf{G}_v}) \to Z(\widehat{\mathbf{H}_v})$ of Γ_{F_v}-modules we obtain a short exact sequences of Γ_{F_v}-modules

$$
0 \to Z(\widehat{\mathbf{G}_v}) \to Z(\widehat{\mathbf{H}_v}) \to Z(\widehat{\mathbf{H}_v})/Z(\widehat{\mathbf{G}_v}) \to 0 \tag{236}
$$

with similar compatibilities as in (234).

We further denote by

$$
\mathrm{inv} : Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}) \to H^1(\Gamma, Z(\widehat{\mathbf{G}})) \tag{237}
$$

and

$$
\mathrm{inv}_v : Z(\widehat{\mathbf{H}_v})/Z(\widehat{\mathbf{G}_v}) \to H^1(\Gamma_v, Z(\widehat{\mathbf{G}_v})) \tag{238}
$$

the connecting homomorphisms associated to (235) and (236) respectively. Under the aforementioned Γ_v-equivariant local-global identifications it's easy to see that inv_v can be identified with with the composition of inv and the localization map $H^1(\Gamma, Z(\widehat{\mathbf{G}})) \to H^1(\Gamma_v, Z(\widehat{\mathbf{G}})$.

With this setup, we can define the Kottwitz group as follows:

Definition IV.1.34. *Let F be a number field and let $\mathbf{G}$ be a reductive group over F. Let $\mathbf{H}$ be a reductive subgroup of G of the same rank. Define the* Kottwitz group $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ *as follows:*

$$
\mathfrak{K}(\mathbf{G}, \mathbf{H}, F) := \left\{ \alpha \in (Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}))^\Gamma : \mathrm{inv}(\alpha) \in \ker^1(\Gamma, Z(\widehat{\mathbf{G}})) \right\} \tag{239}
$$

If $\gamma \in \mathbf{G}(F)$ is semisimple, we denote by $\mathfrak{K}(I_\gamma/F)$ the group $\mathfrak{K}(\mathbf{G}, I_\gamma, F)$.

It will be helpful later to note that our definition of $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ differs from the definition given in [Kot84b] and [Kot90] where, instead, Kottwitz uses the group $\pi_0(\mathfrak{K}(\mathbf{G}, \mathbf{H}, F))$ where $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ is given the Hausdorff topology inheirted from the complex Lie group $Z(\widehat{\mathbf{H}})$.

The definition we have chosen to use is more in line with the later work of Kottwitz and other authors (e.g. see [Shi10]). That said, since we would like to make use of the material in [Kot84b] and [Kot86b] we would like to verify that our two definitions agree when $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic.

Namely, we have the following:

Lemma IV.1.35. *Let F be a number field and $\mathbf{G}$ a reductive group over F such that $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic. If $\mathbf{H}$ is a connected reductive subgroup of $\mathbf{G}$ of the same rank, then $\mathfrak{K}(\mathbf{G},\mathbf{H},F)$ is finite and, in particular, is equal to $\pi_0(\mathfrak{K}(\mathbf{G},\mathbf{H},F))$.*

To prove this, it will be helpful to make the following basic observation:

Lemma IV.1.36. *Let F be a number field and $\mathbf{G}$ a reductive group over F. Let $\mathbf{H}$ be a reductive subgroup of G of the same rank. Let $\mathbf{T}$ be a maximal torus of $\mathbf{H}$. Then, there is a natural inclusion*

$$\mathfrak{K}(\mathbf{G},\mathbf{H},F)\hookrightarrow\mathfrak{K}(\mathbf{G},\mathbf{T},F) \tag{240}$$

Proof. Let us merely observe that, by the proof of Lemma IV.1.32, we have a Γ-equivariant inclusions

$$Z(\widehat{\mathbf{G}})\hookrightarrow Z(\widehat{\mathbf{H}})\hookrightarrow\widehat{\mathbf{T}} \tag{241}$$

which gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z(\widehat{\mathbf{G}})^{\Gamma} \arrow[r] \arrow[d, "\mathrm{id}"'] & Z(\widehat{\mathbf{H}})^{\Gamma} \arrow[r] \arrow[d, hook] & (Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}))^{\Gamma} \arrow[r] \arrow[d, hook] & H^1(\Gamma,Z(\widehat{\mathbf{G}})) \arrow[d] \\
0 \arrow[r] & Z(\widehat{\mathbf{G}})^{\Gamma} \arrow[r] & \widehat{\mathbf{T}}^{\Gamma} \arrow[r] & (\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma} \arrow[r] & H^1(\Gamma,Z(\widehat{\mathbf{G}}))
\end{tikzcd}$$

from where it's clear that we get the desired inclusion $\mathfrak{K}(\mathbf{G},\mathbf{H},F)\hookrightarrow\mathfrak{K}(\mathbf{G},\mathbf{T},F)$. □

From Lemma IV.1.36 the proof of Lemma IV.1.35 follows immediately from the following:

Lemma IV.1.37. *Let F be a number field and $\mathbf{G}$ a reductive group over F. Let $\mathbf{T}$ be a torus in $\mathbf{G}$ containing $Z(\mathbf{G})$ which is elliptic. Then $(\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma}$ is finite.*

Proof. Let us begin by showing that for any torus $\mathbf{S}$ over F there is a natural identification of $\widehat{\mathbf{S}}^{\Gamma}$ and $D(\mathbb{C})$ where D is the diagonalizable $\mathbb{C}$-group with character lattice $X_*(\mathbf{S})_{\Gamma_F}$ (the Γ_F-coinvariants of $X_*(\mathbf{S})$).

Now, we write $\mathbf{G}^{\mathrm{Sc}}$ to denote the simply connected cover of $\mathbf{G}^{\mathrm{ad}}$. Then denote by $\mathbf{T}^{\mathrm{ad}}$ the projection of $\mathbf{T}$ to $\mathbf{G}^{\mathrm{ad}}$ and $\mathbf{T}^{\mathrm{sc}}$ the pre-image of $\mathbf{T}^{\mathrm{ad}}$ under the surjection $\mathbf{G}^{\mathrm{sc}}\to\mathbf{G}^{\mathrm{ad}}$. Then $\mathbf{T}^{\mathrm{ad}}=\mathbf{T}/Z(\mathbf{G})$ and the projection $\mathbf{T}^{\mathrm{sc}}\to\mathbf{T}^{\mathrm{ad}}$ is an isogeny so that we have a Γ_F-equivariant isomorphism

$$X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{Q}}\cong X_*(\mathbf{T}^{\mathrm{sc}})_{\mathbb{Q}}. \tag{242}$$

Taking coinvariants and applying the previous paragraph as well as basic theory of actions of finite groups on $\mathbb{Q}$-spaces, we get

$$X^*(\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F})_{\mathbb{Q}}=X_*(\mathbf{T}^{\mathrm{sc}})_{\Gamma}\otimes\mathbb{Q}\cong X_*(\mathbf{T}^{\mathrm{ad}})_{\Gamma_F}\otimes\mathbb{Q}=X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{Q}}^{\Gamma_F}. \tag{243}$$

Now, $X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{Q}}^{\Gamma_F} = 0$ since $\mathbf{T}^{\mathrm{ad}}$ is anisotropic. Then, a diagonalizable group D is finite if and only if $X^*(D)_{\mathbb{Q}}$ is trivial which implies that $\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F}$ is finite. But $\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F} = (\widehat{\mathbf{T}}^{\mathrm{ad}})^{\Gamma_F} = (\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma_F}$ so this is the desired result. $\square$

IV.1.6 Preservation of properties under Weil restriction

In this appendix we merely collect the verification that several properties of algebraic groups used in this note are preserved under Weil restriction:

Lemma IV.1.38. *Let F/F' be a finite extension. Let H be a reductive gorup over a field F' such that H^{ad} is F'-anisotropic. Then, $(\mathsf{Res}_{F/\mathbb{Q}}H)^{\mathrm{ad}}$ is F-anisotropic.*

Proof. The claim is trivial given Lemma IV.4.20 since we have the equality $(\mathsf{Res}_{F/\mathbb{Q}}H)(F) = H(F')$. $\square$

Lemma IV.1.39. *Let F'/F be an extension of number fields. Let $\mathbf{H}$ be a reductive group over F' which satisfies the Hasse principle. Then, $\mathsf{Res}_{F'/F}\mathbf{H}$ satisfies the Hasse principle.*

Proof. Begin by noting that we have the following commutative diagram

$$\begin{tikzcd}
H^1(F',\mathbf{H}) \arrow[rr] \arrow[ddd, "(1)"', "\wr"] & & \prod_w H^1(F'_w,\mathbf{H}) \arrow[d, "||"] \\
& & \prod_v \prod_w H^1(\mathbf{F}'_w, H) \arrow[dd, "(2)"', "\wr"] \\
& & \\
H^1(F,\mathsf{Res}_{F'/F}\mathbf{H}) \arrow[rr] & & \prod_v H^1(F_v,\mathsf{Res}_{F'/F}\mathbf{H})
\end{tikzcd} \tag{244}$$

The isomorphism in arrow (1) is just Shapiro's lemma. To see the isomorphism in arrow (2) we proceed as follows:

$$\begin{aligned}
H^1(F_v,\mathsf{Res}_{F'/F}\mathbf{H}) &= H^1_{\text{ét}}(F_v,(\mathsf{Res}_{F'/F}\mathbf{H})_{F_v}) \\
&\cong H^1_{\text{ét}}(F_v,\mathsf{Res}_{F'_v/F_v}\mathbf{H}_{F'_v}) \\
&\cong H^1_{\text{ét}}\left(F_v,\prod_{w|v}\mathsf{Res}_{F'_w/F_v}\mathbf{H}_{F'_w}\right) \\
&\cong \prod_{w|v} H^1_{\text{ét}}(F_v,\mathsf{Res}_{F'_w/F_v}\mathbf{H}_{F'_w}) \qquad (245) \\
&\overset{\cong}{(3)} \prod_{w|v} H^1_{\text{ét}}(F'_w, H_{F_w}) \\
&= \prod_{w|v} H^1(F'_w,\mathbf{H})
\end{aligned}$$

where, obviously, the isomorphism labeled (3) is just Shapiro's lemma.

The commutativity of this diagram, and the fact that the vertical maps are isomorphisms, gives an isomorphism

$$\ker^1(F', \mathbf{H}) \cong \ker^1(F, \mathsf{Res}_{F'/F}\mathbf{H}) \tag{246}$$

from where the conclusion follows. $\square$

Lemma IV.1.40. *Let F'/F be an extension of number fields. Let $\mathbf{H}$ be a reductive F'-group such that $\mathbf{H}^{\mathrm{ad}}$ is F'-anisotropic, $\mathbf{H}$ satisfies the Hasse principle, and $\mathbf{H}$ has no relevant global endoscopy. Then, $\mathsf{Res}_{F'/F}\mathbf{H}$ has no relevant global endoscopy.*

Proof. By Proposition I.5.3 it suffices to show that for all maximal F'-tori $\mathbf{T}' \subseteq \mathsf{Res}_{F'/F}\mathbf{H}'$ that the equality

$$Z(\widehat{\mathsf{Res}_{F'/F}\mathbf{H}})^{\Gamma_F} = \widehat{\mathbf{T}'}^{\Gamma_F} \tag{247}$$

holds. Note though that $\mathbf{T}' = \mathsf{Res}_{F'/F}\mathbf{T}$ for some maximal torus $\mathbf{T}$ in H (e.g. see [CGP15, Proposition A.5.15 (2)]). Note now though that since

$$\widehat{\mathbf{T}'} \cong \widehat{\mathbf{T}}^{[F':F]} \tag{248}$$

with Γ_F acting through its quotient $\mathrm{Gal}(F'/F)$ which acts by permutation of the factors, that

$$\widehat{\mathbf{T}'}^{\Gamma_F} = \widehat{\mathbf{T}}^{\Gamma_{F'}} \tag{249}$$

and similarly

$$Z(\widehat{\mathsf{Res}_{F'/F}\mathbf{H}})^{\Gamma_F} = Z(\widehat{\mathbf{H}})^{\Gamma_{F'}} \tag{250}$$

from where the equality follows from Lemma I.5.3 and the fact that $\mathbf{H}$ has no relevant global endoscopy. $\square$

Lemma IV.1.41. *Let F'/F be an extension of fields. Let H be a areductive group over a field F' with H^{der} simply connected. Then, $\mathsf{Res}_{F'/F}H$ has simply connected derived subgroup.*

Proof. Begin by noting that $(\mathsf{Res}_{F'/F}H)^{\mathrm{der}} \cong \mathsf{Res}_{F'/F}H^{\mathrm{der}}$. Note though that we can check derived subgroup over algebraic closure. But

$$(\mathsf{Res}_{F'/F}H^{\mathrm{der}})_{\overline{F}} \cong (H^{\mathrm{der}}_{\overline{F}})^{[F':F]} \tag{251}$$

Since we're in characteristic zero, the fundamental group splits across direct products and so

$$\pi_1^{\text{ét}}\left((H^{\mathrm{der}}_{\overline{F}})^{[F':F]}, \overline{x}\right) \cong \pi_1^{\text{ét}}((H^{\mathrm{der}}_{\overline{F}}), \overline{x})^{[F':F]} = 0 \tag{252}$$

as desired. $\square$

IV.1.7 Some lemmas about transfer

In this subsection we establish several results concerning transferability of conjugacy classes. We begin with the following observation:

Lemma IV.1.42. *Let F be a field of characteristic 0 and let G be a quasi-split group over F. Let $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ be an inner twist. Let T be a torus of G which transfers to G' (in the sense of [Kal16, §3.2]) then for any $\gamma \in T(F)$ the conjugacy class of γ transfers to a conjugacy class in $G'(F)$ (in the sense of [Shi10, §2.3]).*

Proof. By definition there exists some $g \in G(\overline{F})$ such that the map $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}} : T_{\overline{F}} \to G'_{\overline{F}}$ is defined over F. Let T' be the image of T under the descent of $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}}$ to F. Note then that taking $\mathbb{T}_H := T_{\overline{F}}$ and $\mathbb{T} := T'_{\overline{F}}$ as in [Shi10, §2.3] we have that θ can be taken to be $\mathrm{Int}(\psi(g)) \circ \psi$. Then, by definition, γ transfers to a conjugacy class in $G'(F)$ if and only if $\theta(g) \in T'(\overline{F})$ has an element of its associated $G(\overline{F})$-conjugacy class defined over F. But, evidently we can take the image of γ under the descent of $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}}$ to F. The conclusion follows. □

One thing that follows immediately from this is the following:

Corollary IV.1.43. *Let F be a p-adic local field let G be a quasi-split group over F. Let $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ be an inner twist. Let T be an elliptic maximal torus of G. Then, any element $\gamma \in T(F)$ transfers to a conjugacy class in $G'(F)$.*

Proof. This follows immediately by combining Lemma IV.1.42 and [Kot86b, §10] (see also [Kal16, Lemma 3.2.1] □

IV.2 Appendix 2: The trace formula in the anisotropic case and its pseudo-stabilization

In this appendix we record, mostly for the convenience of the reader and to set notation, the Arthur-Selberg trace formula in the compact case or, said differently, for a reductive group $\mathbf{G}$ over $\mathbb{Q}$ such that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic (which is a blanket assumption throughout this assumption assuming throughout this section unless stated otherwise). We will often times assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected to simplify the discussion, but this is rarely strictly necessary.

We then write out the pseudo-stabilization of this trace formula under the assumption that $\mathbf{G}$ has no relevant global elliptic endoscopy (in the sense of §I.5).

IV.2.1 The trace formula in the compact case

In this subsection we recall the Arthur-Selberg trace formula in the case when $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. For the beginning part of this section, one can put no restrictions on $\mathbf{G}$ other than it being reductive.

We begin with the following lemma that will be continually useful in the following:

Lemma IV.2.1. *Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$. Then, the group $\mathbf{G}(\mathbb{A})$ is an internal direct product of $A_{\mathbf{G}}(\mathbb{R})^0$ and $\mathbf{G}(\mathbb{A})^1$. In particular the natural map*

$$[\mathbf{G}] \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0 \tag{253}$$

is an isomorphism of topological measure spaces.

Before we begin the proof, Let us note that we will often times shorten the notation for an element $\mathbf{G}(\mathbb{Q})x$ in $[\mathbf{G}]$ to the notation $[x]$.

Proof. (Lemma IV.2.1) Since $A_{\mathbf{G}}(\mathbb{R})^0$ and $\mathbf{G}(\mathbb{A})^1$ are normal we need to show that the equality $A_{\mathbf{G}}(\mathbb{R})^0\mathbf{G}(\mathbb{A})^1 = \mathbf{G}(\mathbb{A})$ holds and $A_{\mathbf{G}}(\mathbb{R})^0 \cap \mathbf{G}(\mathbb{A})^1$ is trivial. This latter fact is clear. The former follows easily from the decomposition $\mathbf{G} = \mathbf{G}^{\mathrm{der}}Z(\mathbf{G})$ which shows that the natural map $X^*(\mathbf{G}) \to X^*(A_{\mathbf{G}})$ is injective with finite cokernel. The second claim readily follows. $\square$

Because of this lemma we will conflate $[\mathbf{G}]$ with $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0$ and, in particular, call this latter topological measure space (with the measure induced from the Haar measure on $\mathbf{G}(\mathbb{A})$) the adelic quotient.

Let us now set up some of the necessary notation. Namely, let us fix a smooth character $\chi : A_{\mathbf{G}}(\mathbb{R})^+ \to \mathbb{C}$ and let us make the following definition:

Definition IV.2.2. *We denote by $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ the space of functions $\phi : \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ such that $\phi(ax) = \chi(a)\phi(x)$ for all $a \in A_{\mathbf{G}}(\mathbb{R})^0$ and such that $\phi\chi^{-1}$ is square-integrable on $[\mathbf{G}]$.*

Note that combining the fact that $\mathbf{G}(\mathbb{Q}) \cap A_{\mathbf{G}}(\mathbb{R})^0$ is trivial with Lemma IV.2.1 we see that every element $\alpha \in \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ can be written in the form $\alpha = \mathbf{G}(\mathbb{Q})ax$ with $a \in A_{\mathbf{G}}(\mathbb{R})^0$ and $x \in \mathbf{G}(\mathbb{A})^1$ and, moreover, a and $\mathbf{G}(\mathbb{Q})x$ are unique. In particular, the function $(\phi\chi^{-1})(\alpha) := \chi^{-1}(a)\phi(\mathbf{G}(\mathbb{Q})x)$ makes sense as a function $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$. Moreover, it's clear that since $\phi\chi^{-1}$ is $A_{\mathbf{G}}(\mathbb{R})^0$ invariant it descends to a function $[\mathbf{G}] \to \mathbb{C}$ which we also denote $\phi\chi^{-1}$.

Let us now set the following notation:

Definition IV.2.3. *We denote by $\mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ the set of $\mathbb{C}$-linear combinations of functions $f = f_\infty f^\infty : \mathbf{G}(\mathbb{A}) \to \mathbb{C}$ where:*

1. *$f^\infty : \mathbf{G}(\mathbb{A}_f) \to \mathbb{C}$ is locally constant and compactly supported.*

2. *$f_\infty : \mathbf{G}(\mathbb{R}) \to \mathbb{C}$ is smooth, satisfies $f(ax) = \chi(a)^{-1}f(x)$ for all $a \in A_{\mathbf{G}}(\mathbb{R})^0$, and for which $f\chi$ is compactly supported as a function on $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0$.*

If $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ note that we get a compactly supported function $f\chi : \mathbf{G}(\mathbb{A})^1 \to \mathbb{C}$ defined by $(f\chi)(ax) := f(x)$ where $a \in A_{\mathbf{G}}(\mathbb{R})^0$ and $x \in \mathbf{G}(\mathbb{A})^1$ (again using Lemma IV.2.1).

We now make a definition of the operators $R_\chi(f)$ and $R(f\chi)$ for an element $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. Namely:

Definition IV.2.4. *The* right convolution operator $R_\chi(f)$ *on* $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ *is defined by taking $\phi \in L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ to*

$$R_\chi(f)(\phi)(\mathbf{G}(\mathbb{Q})x) := \int_{\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0} f(g)\phi(\mathbf{G}(\mathbb{Q})xg)\, dg \tag{254}$$

which is well-defined since f and ϕ transform by inverse characters and f is compactly supported on $\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^+$.

We also define the operator $R(f\chi^{-1})$ on $L^2([\mathbf{G}])$ as

$$R(f\chi)(\psi)([x]) := \int_{\mathbf{G}(\mathbb{A})^1} (f\chi)(g)\psi([xg])\, dg \tag{255}$$

We then have the following elementary observation:

Lemma IV.2.5. *We have a natural isomorphism of $\mathbb{C}$-spaces*

$$L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) \xrightarrow{\approx} L^2([\mathbf{G}]) : \phi \mapsto \phi\chi^{-1} \tag{256}$$

which is equivariant for the natural $\mathbf{G}(\mathbb{A})^1$-action on both sides and such that

$$R_\chi(f)(\phi) = R(f\chi)(\phi\chi^{-1}) \tag{257}$$

Proof. We can define an inverse of the above map by pulling back a function $\phi \in L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^+)$ along the quotient map

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0, \tag{258}$$

and twisting by χ.

Now, we have

$$R_\chi(f)(\phi)(\mathbf{G}(\mathbb{Q})x) = \int_{\mathbf{G}(\mathbb{A})/A_G(\mathbb{R})^0} f(g)\phi(\mathbf{G}(\mathbb{Q})xg)dg \tag{259}$$

$$= \int_{\mathbf{G}(\mathbb{A})^1} (f\chi)(g)(\phi\chi^{-1})(xg)dg \tag{260}$$

$$= R(f\chi)(\phi\chi^{-1})(x). \tag{261}$$

from where the lemma follows. □

From this point on we assume that $\mathbf{G}^{\text{ad}}$ is $\mathbb{Q}$-anisotropic and $\mathbf{G}^{\text{der}}$ is simply connected. This has the benefit of implying that $I_\gamma = Z_{\mathbf{G}}(\gamma)$ for all $\gamma \in \mathbf{G}(\mathbb{Q})$ and thus $a(\gamma) = 1$ for all semi-simple $\gamma \in \mathbf{G}(\mathbb{Q})$.

Let us now appeal to the following result which justifies our terminology of calling the situation when $\mathbf{G}^{\text{ad}}$ is $\mathbb{Q}$-anisotropic the 'compact case':

Theorem IV.2.6 (Borel, Harish-Chandra)**.** *Let $\mathbf{H}$ be a reductive group over $\mathbb{Q}$. Then, the space $[\mathbf{H}]$ is compact if and only if $\mathbf{H}^{\text{ad}}$ is $\mathbb{Q}$-anisotropic.*

Proof. The desired result is contained in [Con12a, §A.5]. Note, in particular, that since $\mathbf{H}$ was assumed reductive that [Con12a, Lemma A.5.2] shows that conditions a) and b) are equivalent to $\mathbf{H}^{\text{ad}}$ being $\mathbb{Q}$-anisotropic. □

Note then that we have the following well-known result:

Theorem IV.2.7. *For any function $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ the operator $R(f\chi)$ on $L^2([\mathbf{G}])$ is trace class. Moreover, there is a decomposition*

$$L^2([\mathbf{G}]) = \bigoplus_{\pi' \in \Pi(\mathbf{G}(\mathbb{A})^1)} m(\pi')\pi' \tag{262}$$

where $\Pi(\mathbf{G}(\mathbb{A})^1)$ denotes the set of irreducible unitary $\mathbf{G}(\mathbb{A})^1$-subrepresentations and $m(\pi')$ is some integer (possibly zero).

Proof. This is a classical, and well-known result that follows from easy function analysis since $[\mathbf{G}]$ is compact. For example, see [Whi, §3]. □

From this we deduce the following:

Corollary IV.2.8. *The operator $R_\chi(f)$ on the space $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ is trace class and there is a decomposition*

$$L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) = \bigoplus_{\pi \in \Pi_\chi(\mathbf{G}(\mathbb{A}))} m(\pi)\pi \tag{263}$$

where $\Pi_\chi(\mathbf{G}(\mathbb{A}))$ denotes the set of irreducible unitary $\mathbf{G}(\mathbb{A})$-representations acting by the character χ on $A_{\mathbf{G}}(\mathbb{R})^+$ and $m(\pi)$ is some integer (possibly zero).

Proof. The fact that $R_\chi(f)$ is trace class follows from the map constructed in IV.2.5. The decomposition follows from this map as well as the fact that $A_{\mathbf{G}}(\mathbb{R})^0$ is central in $\mathbf{G}(\mathbb{A})$, hence extending $\mathbf{G}(\mathbb{A})^1$ representations to $\mathbf{G}(\mathbb{A})$ via a character of $A_{\mathbf{G}}(\mathbb{R})^0$ does not affect the decomposition into irreducible representations. □

We would now like to state the Arthur-Selberg trace formula in this context. Before we do this, it's useful to note the following trivial finiteness result.

Lemma IV.2.9. *Let $\mathbf{H}$ be a reductive group over a global field F and let $C \subset \mathbf{H}(\mathbb{A}_F)$ a compact subset. Then $\mathbf{H}(F) \cap C$ is finite.*

Proof. This is essentially trivial. It suffices to show that $\mathbf{H}(F) \cap C$ is discrete and compact. The group $\mathbf{H}(F) \subset \mathbf{H}(\mathbb{A}_F)$ is discrete, therefore so is $\mathbf{H}(F) \cap C$. But, $\mathbf{H}(F)$ is also closed in $\mathbf{H}(\mathbb{A}_F)$ (as any discrete subgroup of a Hausdorff group is closed) and thus $\mathbf{H}(F) \cap C$, being a closed subset of C, is also compact. The conclusion follows. $\square$

From this we deduce the following:

Corollary IV.2.10. *Let $\mathbf{H}$ be a reductive group over a global field F. Suppose that $C \subseteq \mathbf{H}(\mathbb{A})$ is such that its projection to $\mathbf{H}(\mathbb{A})/A_{\mathbf{H}}(\mathbb{R})^0$ is compact. Then, C meets finitely many $\mathbf{H}(F)$-conjugacy classes.*

Proof. Note that since $\mathbf{H}(F)$-conjugacy classes are separated by the natural map $\mathbf{H}(F) \to \mathbf{H}^{\mathrm{ad}}(F)$ it suffices to show that the projection of C along the projection $\mathbf{H}(\mathbb{A}) \to \mathbf{H}^{\mathrm{ad}}(\mathbb{A})$ intersects only finitely many $\mathbf{H}^{\mathrm{ad}}(F)$ conjugacy classes. But, note that C has compact image in $\mathbf{H}^{\mathrm{ad}}(\mathbb{A})$, since the map $\mathbf{H}(\mathbb{A}) \to \mathbf{H}^{\mathrm{ad}}(\mathbb{A})$ factors through the map $\mathbf{H}(\mathbb{A}) \to \mathbf{H}(\mathbb{A})/A_{\mathbf{H}}(\mathbb{R})^0$, and thus the claim follows easily from the previous lemma. $\square$

Let us now assume that $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. We then define, as in the notation at the beginning of this article, the notion of an orbital integral:

Definition IV.2.11. *Let $\gamma \in \mathbf{G}(\mathbb{Q})$ be given. Then, the* orbital integral *of f relative to γ is the following:*

$$O_\gamma(f) := \int_{I_\gamma(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} f(g^{-1}\gamma g)\, dg \tag{264}$$

This integral converges because of our assumption that f lies in the set $\mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ (and, in particular, has compact support modulo $A_{\mathbf{G}}(\mathbb{R})^0$).

Let us also note that $[I_\gamma]$ is compact since I_γ, being a closed subgroup of $\mathbf{G}$, also satisfies $I_\gamma/Z(I_\gamma)$ is $\mathbb{Q}$-anisotropic. Thus, $v_\gamma := \mathrm{vol}([I_\gamma])$, which is equal (by definition) to $\tau(I_\gamma)$, is finite. Note that both $O_\gamma(f)$ and $\mathrm{vol}([I_\gamma])$ only depend on the conjugacy class $\{\gamma\}$ in $\mathbf{G}(\mathbb{Q})$.

Definition IV.2.12. *For $(\pi, V) \in \Pi_\chi(\mathbf{G}(\mathbb{A}))$ and $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$, we define the* trace $\mathrm{tr}(f|\pi)$ *to be the trace of the operator $\pi(f)$ on V given by*

$$\pi(f) := v \mapsto \int_{\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0} f(g)\pi(g)v dg. \tag{265}$$

Let us note that any element of $\Pi_\chi(\mathbf{G}(\mathbb{A}))$ is admissible (as follows from Harish-Chandra's finiteness results as in [BJ79, THeorem 1.7]), and thus this trace is a well-defined complex number.

Before we finally state the trace formula, we record the following fact implicitly used in the sequel:

Lemma IV.2.13. *Let $\mathbf{H}$ be a reductive group over $\mathbb{Q}$. Suppose that γ is an elliptic element of $\mathbf{H}(\mathbb{Q})$. Then $I_\gamma(\mathbb{A})^1 = I_\gamma(\mathbb{A}) \cap \mathbf{H}(\mathbb{A})^1$.*

Proof. First note that we really do need the assumption that γ is elliptic as the example in [AEK05, §4, pg20] indicates.

To prove the lemma, we first show that $X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}} = X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}}$. Indeed, we have isogenies

$$Z(\mathbf{H}) \to \mathbf{H}^{\mathrm{ab}}, \qquad Z(I_\gamma) \to I_\gamma^{\mathrm{ab}} \tag{266}$$

and hence isomorphisms

$$X^*_{\mathbb{Q}}(Z(\mathbf{H}))_{\mathbb{Q}} \cong X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}}, \qquad X^*_{\mathbb{Q}}(Z(I_\gamma))_{\mathbb{Q}} \cong X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}} \tag{267}$$

Additionally, since γ is elliptic, we have

$$X^*_{\mathbb{Q}}(Z(I_\gamma)) = X^*_{\mathbb{Q}}(Z(\mathbf{H})) \tag{268}$$

Putting these isomorphisms together, gives the desired equality.

Now, we then have

$$\begin{aligned} I_\gamma(\mathbb{A})^1 &:= \{h \in I_\gamma(\mathbb{A}) : |\chi(h)| = 1\, \forall \chi \in X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}}\} & (269)\\ &= \{h \in I_\gamma(\mathbb{A}) : |\chi(h)| = 1\, \forall \chi \in X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}}\} & (270)\\ &= I_\gamma(\mathbb{A}) \cap \mathbf{H}(\mathbb{A})^1 & (271)\end{aligned}$$

as desired. □

We then have the following:

Theorem IV.2.14. *Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Then, for any function $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ we have an equality*

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\mathrm{s.s.}}} v_\gamma O_\gamma(f) = \mathrm{tr}(R_\chi(f)) \tag{272}$$

Let us note that by Corollary IV.2.10 the sum on the left-hand side of (272) is a finite sum, and thus is convergent. The right-hand side of (272) is convergent since $R_\chi(f)$ is trace class by Corollary IV.2.8.

Proof. (Theorem IV.2.14) This follows from the discussion in [AEK05, §1.1]. Namely, from the discussion therein, since $[\mathbf{G}]$ is compact we get an equality of $\mathrm{tr}(R(f\chi))$ with

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\mathrm{s.s.}}} \mathrm{vol}(I_\gamma(\mathbb{Q}) \backslash I(\mathbb{A})^1_\gamma) \int_{I(\mathbb{A})^1_\gamma \backslash \mathbf{G}(\mathbb{A})^1} (f\chi)(g^{-1}\gamma g)\, dg \tag{273}$$

But, from Lemma IV.2.5 we know that $\mathrm{tr}(R_\chi(f)) = \mathrm{tr}(R(f\chi))$. Moreover, it's easy to see that (273) agrees with the left hand side of (272) for $f\chi$ in place of f with the only subtle point being the contents of Lemma IV.2.13. The conclusion follows. □

Finally, we use Corollary IV.2.8 to deduce:

Corollary IV.2.15. *Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Then, for any $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ we have an equality*

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f \mid \pi) \tag{274}$$

where $\Pi_\chi(\mathbf{G})$ and $m(\pi)$ are as in Corollary IV.2.8.

IV.2.2 Pseudo-stabilization

Our goal is now to rewrite Corollary IV.2.15 in terms of stable orbital integrals. Namely, we aim to prove the following:

Proposition IV.2.16. *Suppose that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$ anisotropic and $\mathbf{G}$ has no relevant global elliptic endoscopy (in the sense of §I.5). Let $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. Then,*

$$\tau(\mathbf{G}) \sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} SO_\gamma(f) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f \mid \pi) \tag{275}$$

where $m(\pi)$ is as in Corollary IV.2.15.

To prove this, we will manipulate the left hand side of (274) into the left hand side of (275). We will mainly be following the material in [Kot86b, §6].

To start, let us first write

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\{\gamma_0\} \in \{\mathbf{G}\}^{\text{s.s.}}_s} \sum_{\{\gamma\} \in S(\gamma_0)} v_\gamma O_\gamma(f) \tag{276}$$

We now have the following

Lemma IV.2.17 ([Kot84b])**.** *Let $\mathbf{H}$ and $\mathbf{H}'$ reductive groups over $\mathbb{Q}$ which are inner forms. Then, $\tau(\mathbf{H}) = \tau(\mathbf{H}')$.*

Proof. By [Kot84b, (5.1.1)], (since $\tau(\mathbf{H}_{\mathrm{sc}}) = 1$ by the resolution of the Tamagawa conjecture by Kottwitz in [Kot88]) we have

$$\tau(\mathbf{H}) = |\pi_0(Z(\widehat{\mathbf{H}})^\Gamma| \cdot |\ker^1(F, Z(\widehat{\mathbf{H}}))|^{-1}. \tag{277}$$

Since we have a Γ-equivariant isomorphism $\widehat{\mathbf{H}} \cong \widehat{\mathbf{H}'}$, this formula immediately implies the desired result. □

Hence, we see that $v_\gamma = v_{\gamma_0}$ for all $\{\gamma\} \in S(\gamma_0)$. Thus, the above becomes

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\{\gamma_0\} \in \{\mathbf{G}\}^{\text{s.s.}}_s} v_{\gamma_0} \sum_{\{\gamma\} \in S(\gamma_0)} O_\gamma(f) \tag{278}$$

To continue, we recall the following lemma of Kottwitz (see §IV.1.5 for notation concerning the Kottwitz group):

Lemma IV.2.18 (Kottwitz)**.** *Let* $\mathbf{H}$ *be a reductive group over a number field* F*. Let* $\gamma_0 \in \mathbf{H}(F)$ *be a given semi-simple element. Then, for a given semi-simple element* $(\gamma_v) = \gamma \in \mathbf{H}(\mathbb{A})$ *such that for all places* v*, we have* $\gamma_v \sim_s \gamma_{0v}$ *one has that* $\gamma \sim \gamma'$ *for some* $\gamma' \in \mathbf{H}(F)$ *if and only if the equality holds*

$$\sum_v \mathrm{obs}(\gamma_0, \gamma_v) \mid_{\mathfrak{K}(I_\gamma/F))} = 0 \tag{279}$$

where both sides are considered as elements of $\mathfrak{K}(I_\gamma/F)$*. Moreover, if there exist such a* γ' *then the number of such* γ' *(up to* $\mathbf{H}(F)$*-conjugacy) is the quantity* $|\mathfrak{K}(I_\gamma/F)|\tau(\mathbf{H})v_{\gamma_0}^{-1}$*.*

Proof. For the first claim see [Kot86b, Theorem 6.6]. For the second claim see the discussion succeeding Equation (9.6.3) on page 394 and the discussion preceding (9.6.5) on page 395 noting, again, that the resolution of the Tamagawa conjecture by Kottwitz in [Kot88] shows that $\tau_1(\mathbf{M}) = \tau(\mathbf{M})$ for any reductive group $\mathbf{M}$ over $\mathbb{Q}$. □

In particular, we see that since $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic and $\mathbf{G}$ has no relevant global endoscopy we see that the following holds:

Corollary IV.2.19. *Let* $\gamma_0 \in \mathbf{G}(F)$ *be a given semi-simple element. Then, for a given semi-simple* $(\gamma_v) = \gamma \in \mathbf{G}(\mathbb{A})$ *such that for all places* v*, we have* $\gamma_v \sim_s \gamma_{0v}$ *one has that* $\gamma \sim \gamma'$ *for some* $\gamma' \in G(F)$*. Moreover, the number of such* γ' *(up to* $G(F)$*-conjugacy) is* $\tau(G)v_{\gamma_0}^{-1}$*.*

From this we see that we can rewrite (278) as follows:

$$\sum_{\{\gamma\}\in\{\mathbf{G}\}^{\mathrm{s.s.}}} v_\gamma O_\gamma(f) = \tau(\mathbf{G}) \sum_{\{\gamma_0\}\in\{\mathbf{G}\}_s^{\mathrm{s.s.}}} \sum_{\gamma\in S_{\mathbb{A}}(\gamma_0)} O_\gamma(f) \tag{280}$$

where $S_{\mathbb{A}}(\gamma_0)$ are the $\mathbf{G}(\mathbb{A})$-conjugacy classes which are stably $\mathbf{G}(\mathbb{A})$-conjugate to $\{\gamma_0\}$. Proposition IV.2.16 then follows considering the term on the right hand side is almost the definition of the term on the left hand side of (275). In particular, we see that in this case, $e(\gamma) = 1$ because at each place v, we have $\gamma_v \sim \gamma'$ for some $\gamma' \in \mathbf{G}(F)$, so that $e(\gamma) = e(I_{\gamma'}) = 1$ from which the claimed equality holds.

IV.3 Appendix 3: Base change for unitary groups

We record here the version of base change necessary for our purposes. We are essentially following the results in [Lab09].

For this appendix we fix a CM number field $\mathbf{E}$ and let $\mathbf{F}$ be its maximal real subfield. We assume that $\mathbf{F} \supsetneq \mathbb{Q}$. Let us also fix an integer $n \geqslant 1$ and let $\mathbf{U}$ be an inner form of $U_{E/F}(n)^*$. We then set $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}}\mathbf{U}$ and $\mathbf{H} := \mathsf{Res}_{E/\mathbb{Q}}\mathrm{GL}_{n,E}$. We fix a cofinite set S_{unram} of primes p of $\mathbb{Q}$ over which

$\mathbf{G}$ is unramified, and for each $p \in S_{\text{unram}}$ we fix a hyperspecial subgroup $K_{0,p} \subseteq \mathbf{G}(\mathbb{Q}_p)$.

Next, let us fix an automorphic representation π of $\mathbf{U}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A})$. We then denote denote by $S_{\text{ram}}(\pi)$ the union of the complement of S_{unram} and the finitely many $p \in S_{\text{unram}}$ for which π_p is ramified relative to $K_{0,p}$.

For every prime $p \notin S_{\text{ram}}(\pi)$ let us note that we have an unramified base change map

$$\mathrm{BC}_p : \left\{ \begin{matrix} \text{Irreducible and smooth} \\ K_{0,p}\text{-unramified} \\ \text{representations of } \mathbf{G}(\mathbb{Q}_p) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Irreducible and smooth} \\ K'_{0,p} - \text{unramified} \\ \text{representations of } \mathbf{H}(\mathbb{Q}_p) \end{matrix} \right\} \tag{281}$$

(where $K'_{0,p}$ is the unique hyperspecial subgroup of $\mathbf{H}(\mathbb{Q}_p)$) as in [Mın11, §2.7] (see also [Mın11, §4.1]).

With this setup, we then have the following result:

Theorem IV.3.1 ([Lab09, Corollaire 5.3])**.** *Fix ξ to be a regular algebraic representation of $\mathbf{G}_{\mathbb{C}}$. Then, there exists a map*

$$\mathrm{BC} : \left\{ \begin{matrix} \textit{Irreducible discrete} \\ \textit{automorphic representations} \\ \textit{of } U_{E/F}(V)(\mathbb{A}_F) \textit{ such that} \\ \pi_\infty \textit{ is } \xi\textit{-cohomological} \end{matrix} \right\} \to \left\{ \begin{matrix} \textit{Irreducible discrete} \\ \textit{automorphic representations} \\ \textit{of } \mathrm{GL}_n(\mathbb{A}_E) \end{matrix} \right\}$$

such that for all primes $p \notin S_{\text{ram}}(\pi)$ we have that

- $\mathrm{BC}(\pi)_p = \mathrm{BC}_p(\pi_p)$.
- $\mathrm{BC}(\pi)^\vee \cong \mathrm{BC}(\pi) \circ c$ *(where c is the conjugation operator corresponding to the non-trivial element of* $\mathrm{Gal}(E/F)$*).*
- *The infinitesimal character of* $\mathrm{BC}(\pi)_\infty$ *is* $(\xi \otimes \xi)^\vee$*.*

IV.4 Appendix 4: Unitary groups

In this appendix we recall the basic theory of unitary groups, their local-to-global construction, and when such groups have no relevant endoscopy as in §I.5.

IV.4.1 Decomposition of the forms of a split group

Before we begin discussing unitary groups in earnest, it will be helpful to first recall the decomposition of the forms of a split group G into classes corresponding to inner and outer forms.

To begin, let F be any field, assumed perfect for convenience, and let G be a reductive group over F. Recall then the following well-known definition:

Definition IV.4.1. *A* form *or* twist *of G is an algebraic group H over F such that $H_{\overline{F}}$ is isomorphic to $G_{\overline{F}}$. An isomorphism of forms is merely an isomorphism of algebraic groups over F.*

Let us denote by $\mathsf{Form}(G)$ the set of (isomorphism classes of) forms of G. The set $\mathsf{Form}(G)$ is a pointed set with identity element the isomorphism class of G itself.

We recall the cohomological characterization of the pointed set $\mathsf{Form}(G)$. The group functor sending an F-algebra R to the group $\mathrm{Aut}(G_R)$ of R-automorphisms of G_R is representable by a separated and smooth group scheme denoted $\mathrm{Aut}(G)$ (e.g. see [Con14, Theorem 7.1.9]). Note then that associated to this group scheme $\mathrm{Aut}(G)$ there are two pointed sets. The étale cohomology set $H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G))$ (as on [Mil80, Page 122]) and the Galois cohomology set $H^1(F, \mathrm{Aut}(G))$.

We have a natural map of pointed sets

$$\mathsf{Form}(G) \to H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \tag{282}$$

and a natural map

$$\mathsf{Form}(G) \to H^1(F, \mathrm{Aut}(G)) \tag{283}$$

defined as follows. The first map takes a twist H of G to the $\mathrm{Aut}(G)$-torsor $\underline{\mathrm{Isom}}(H, G)$ (where, here, we have used the identification given by [Mil80, Proposition 4.6]). The second map is defined as follows. Let H be an element of $\mathsf{Form}(G)$ and let $f : G_{\overline{F}} \to H_{\overline{F}}$ be an isomorphism. Then, the association

$$\iota_f : \sigma \mapsto \iota_f(\sigma) := f^{-1} \circ \sigma_H \circ f \circ \sigma_G^{-1} \tag{284}$$

defines a map $\iota_f : \Gamma_F \to Z^1(F, \mathrm{Aut}(G))$. Differing choices of f or H (within the same F-isomorphism class) define cohomologous elements of $Z^1(F, \mathrm{Aut}(G))$ and thus we get a well-defined map as in (283).

We then have the following well-known proposition:

Proposition IV.4.2. *There is a commuting triangle of isomorphisms of pointed sets*

$$\begin{tikzcd} \mathsf{Form}(G) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \arrow[dl] \\ H^1(F, \mathrm{Aut}(G)) & \end{tikzcd} \tag{285}$$

where the two arrows emanating from $\mathsf{Form}(G)$ are (282) and (283), and the remaining arrow is the one from [Sta18, Tag03QQ].

Proof. The proof of the bijectivity of the maps (282) and (283) follows easily from the fact that affine morphisms satisfy effective descent (e.g. see [Ser13, §1.3, Chapter III]). The commutivity of the diagram is easy and left to the reader. □

We would like to refine the set of forms of G by decomposing it into its constituents corresponding to whether a form is so-called *inner*. Namely, we make the following well-known definition:

Definition IV.4.3. *An* inner twist *of a group G is a pair (H,ξ) where H is an algebraic group over F and $\xi : G_{\overline{F}} \to H_{\overline{F}}$ is an isomorphism such that $\iota_\xi(\sigma)$ is an inner automorphism of $G_{\overline{F}}$ (i.e. conjugation by some element of $G(\overline{F})$) for every $\sigma \in \Gamma_F$. Two inner twists (H,ξ) and (H',ξ') are* equivalent *if there exists an isomorphism $\phi : H \to H'$ such that $\phi_{\overline{F}} \circ \xi = \mathrm{Int}(h') \circ \xi'$ for some $h' \in H(\overline{F})$.*

The equivalence classes of inner twists of G form a pointed set denoted $\mathsf{InnTwist}(G)$.

We can also classify inner twists of G cohomologically. To do this, begin by noting that we have a natural map of algebraic groups $G^{\mathrm{ad}} \to \mathrm{Aut}(G)$. Indeed, it suffices to give a map $G \to \mathrm{Aut}(G)$ which annihilates $Z(G)$. This map, on R-points, takes an R-point $g \in G(R)$ to the the obvious associated inner automorphism of G_R which is an element of $\mathrm{Aut}(G_R) = \mathrm{Aut}(G)(R)$. From this we obtain a maps of pointed sets

$$H^1_{\text{ét}}(\mathrm{Spec}(F), G^{\mathrm{ad}}) \to H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \tag{286}$$

and

$$H^1(F, G^{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G)) \tag{287}$$

Notice that we also have a natural map

$$\mathsf{InnTwist}(G) \to \mathsf{Form}(G) \tag{288}$$

given by sending (H,ξ) to H.

Note that we also have a map of pointed sets

$$\mathsf{InnTwist}(G) \to H^1(F, G^{\mathrm{ad}}) \tag{289}$$

given by associating to (H,ξ) the element $\iota_\xi \in Z^1(F, G^{\mathrm{ad}})$. Again, one can check that changing (H,ξ) within its equivalence class corresponds to a cohomologous cocycle and thus we get a well-defined map as in (289).

We then have the following (also well-known) proposition:

Proposition IV.4.4. *The following diagram of maps of pointed sets is commutative with the horizontal arrows being isomorphisms*

$$\begin{tikzcd} \mathsf{InnTwist}(G) \arrow[r] \arrow[d] & H^1(F, G^{\mathrm{ad}}) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathrm{Spec}(F), G^{\mathrm{ad}}) \arrow[d] \\ \mathsf{Form}(G) \arrow[r] & H^1(F, \mathrm{Aut}(G)) \arrow[r] & H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \end{tikzcd} \tag{290}$$

where all maps are defined as before this proposition.

Now, the map $\mathsf{InnTwist}(G) \to \mathsf{Form}(G)$ needn't be injective, and we denote by $\mathsf{InnForm}(G)$ its image and call such forms (in the image) *inner forms* of G. Evidently $\mathsf{InnForm}(G)$ can be a proper subset of $\mathsf{Form}(G)$. But, while not every form of G is an inner form, there is a partition of the forms of G in to groupings of the inner forms of certain special forms of G. We now elaborate on this point. While it is not strictly necessary, we assume from this point out that $\underline{G \text{ is split}}$. To this end, we also fix a pair (B, T) consisting of a Borel subgroup B and a split maximal subtorus T of B. We denote the triple (G, B, T) by $\mathcal{P}$.

Begin by recalling that a reductive group H over F is *quasi-split* if it possesses an F-rational Borel subgroup (i.e. a subgroup B of H such that $B_{\overline{F}}$ is a maximal smooth connected solvable subgroup of $H_{\overline{F}}$). We denote the set of (isomorphism classes of) quasi-split forms of G by $\mathsf{QS}(G)$ and thus, by definition, we have an inclusion $\mathsf{QS}(G) \subseteq \mathsf{Form}(G)$. These quasi-split forms of G are the previously alluded to 'special forms' for which every form of G will be an inner form of.

Before we state the decomposition of $\mathsf{Form}(G)$ in terms of these quasi-split forms, we explain how to cohomologically classify the subset $\mathsf{QS}(G)$ of $\mathsf{Form}(G)$. To begin, note that the inclusion of G^{ad} into $\mathrm{Aut}(G)$ has normal image and thus we can form the quotient group scheme which we denote $\mathrm{Out}(G)$. This group scheme is constant, and is finite whenever $Z(G)$ has rank at most 1 (e.g. see [Con14, Proposition 7.1.9]). Note that by definition we have the defining short exact sequence

$$1 \to G^{\mathrm{ad}} \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{291}$$

which gives rise to the diagram

$$\begin{tikzcd} \mathrm{Out}(G)(F) \arrow[r] & H^1(F, G^{\mathrm{ad}}) \arrow[r] \arrow[d] & H^1(F, \mathrm{Aut}(G)) \arrow[r, "\mathrm{cl}"] \arrow[d] & H^1(F, \mathrm{Out}(G)) \\ & \mathsf{InnTwist}(G) & \mathsf{Form}(G) & \end{tikzcd} \tag{292}$$

where the verital maps are bijections and the horizontal maps form an exact sequence of pointed sets. Moreover, we have an idenficiation

$$H^1(F, \mathrm{Out}(G)) = \mathrm{Hom}_{\mathrm{cont.}}(\Gamma_F, \mathrm{Out}(G)(\overline{F}))/\sim \tag{293}$$

where $\sim$ denotes conjugation by $\mathrm{Out}(G)(\overline{F})$. One also has a natural identification of $\mathrm{Out}(G)(\overline{F})$ with the group of automorphisms of the based root datum associated to (G, B, T) (e.g. see [Con14, §1.5] as well as [Con14, Theorem 7.1.9]).

Let us denote by $\mathrm{Aut}(\mathcal{P})$ the subpresheaf of $\mathrm{Aut}(G)$ consisting of those automorphisms preserving $\mathcal{P}$ (i.e. preserving B and T). Note then that we get a natural map

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Aut}(G)) \tag{294}$$

coming from this inclusion.

We then have the following cohomological classification of $\mathsf{QS}(G)$:

Proposition IV.4.5. *The natural map*

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Aut}(G)) \tag{295}$$

is injective with image $\mathsf{QS}(G)$*. Moreover, the natural map*

$$\mathsf{QS}(G) \to H^1(F, \mathrm{Out}(G)) \tag{296}$$

is a bijection. Thus, we have natural bijections

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \xrightarrow{\approx} \mathsf{QS}(G) \xrightarrow{\approx} H^1(F, \mathrm{Out}(G)) \tag{297}$$

Proof. Let us begin by showing that the image of the map in (295) is precisely $\mathsf{QS}(G)$. To do this, let ι is a cocycle of $\mathrm{Aut}(G)(\overline{F})$ with corresponding form H. Suppose now that ι lies in the image of $H^1(F, \mathrm{Aut}(\mathcal{P})$. Then, ι also gives rise (by restriction) to a cocycle in $H^1(F, \mathrm{Aut}(B))$ and thus, by definition, B descends to a form B' of B over F. Since we obtained the cocycle of $H^1(F, \mathrm{Aut}(B))$ by restriction of a cocycle in $H^1(F, \mathrm{Aut}(G))$ we see that we have an embedding $B' \hookrightarrow H$. It's not hard then to see that the image of this B' is a Borel subgroup of H, and thus H is quasi-split.

Suppose now that $H \in \mathsf{QS}(G)$ and fix a pair (B', T') of an F-rational Borel subgroup of H and a maximal torus T' contained in B'. Select an isomorphism $f : G_{\overline{F}} \to H_{\overline{F}}$. Note that by standard algebraic group theory the pair $(f^{-1}(B'_{\overline{F}}), f^{-1}(T'_{\overline{F}}))$ must be conjugate to the pair $(B_{\overline{F}}, T_{\overline{F}})$ by some element $g \in G(\overline{F})$. Note that H corresponds to the cocycle ι_f in $H^1(F, \mathrm{Aut}(G))$. Note then that ι_f is cohomologous to the cocycle $\iota' : \sigma \mapsto g\iota_f(\sigma)\sigma(g)^{-1}$. But, note that ι' (by construction) lands in the image of $H^1(F, \mathrm{Aut}(\mathcal{P}))$ as desired.

If we can show that the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Out}(G))$ is an isomorphism then, since the diagram

$$\begin{tikzcd} H^1(F, \mathrm{Aut}(\mathcal{P})) \arrow[r] \arrow[d] & \mathsf{QS}(G) \arrow[dl] \\ H^1(F, \mathrm{Out}(G)) & \end{tikzcd} \tag{298}$$

commutes the injectivity of $H^1(F, \mathrm{Aut}(\mathcal{P}))$ and the bijectivity of the map $\mathsf{QS}(G) \to H^1(F, \mathrm{Out}(G))$ will follow. Thus, we focus on this.

Let us note that the map $\mathrm{Aut}(\mathcal{P}) \to \mathrm{Out}(G)$ is split (by any pinning of the triple (G, B, T)) and thus so is the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to$

$H^1(F, \mathrm{Out}(G))$. This shows that the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Out}(G))$ is surjective. To show the map is injective note that we have a short exact sequence of group schemes

$$1 \to T/Z(G) \to \mathrm{Aut}(\mathcal{P}) \to \mathrm{Out}(G) \to 1 \tag{299}$$

and thus (by the twisting trick of [Ser13, I, §5.7]) it suffices to show that for all $\mathrm{Out}(G)(\overline{F})$-valued cocycles a one has that $H^1(F, (T/Z(G))_a) = 0$. But, since T is split and the action of a on $X^*(T/Z(G))$ is by permutation of roots, we see that $(T/Z(G))_a$ is an induced torus, and thus the vanishing follows from Shapiro's lemma and Hilbert's theorem 90. $\square$

As a final observation, we give a decomposition of $\mathsf{Form}(G)$ into inner forms of the quasi-split forms of G. Namely, we have the following:

Proposition IV.4.6. *There is a decomposition*

$$\mathsf{Form}(G) = \bigsqcup_{H_0 \in \mathsf{QS}(G)} \mathsf{InnForm}(H_0) \tag{300}$$

Proof. Let us note that we have the exact sequence

$$1 \to G^{\mathrm{ad}} \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{301}$$

which gives rise to the exact sequence

$$H^1(F, G^{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G)) \xrightarrow{p} H^1(F, \mathrm{Out}(G)) \tag{302}$$

Then, clearly, we have a decomposition

$$H^1(F, \mathrm{Aut}(G)) = \bigsqcup_{a \in H^1(F, \mathrm{Aut}(G)} p^{-1}(a) \tag{303}$$

But, by the contents of [Ser13, I, §5.5] we know that $p^{-1}(a)$ is identified of a quotient of $H^1(F, G_a^{\mathrm{ad}})$. But, it's not hard to see that if a corresponds to $H \in \mathsf{QS}(G)$ by Proposition IV.4.5 then $G_a^{\mathrm{ad}} = H^{\mathrm{ad}}$, and the conclusion follows. $\square$

The above decomposition gives us a map $\mathsf{Form}(G) \to \mathsf{QS}(G)$. For an element H of $\mathsf{Form}(G)$ we denote by H^*, an element of $\mathsf{QS}(G)$, the image of H under this map. For a split group G over F we call an element H of $\mathsf{Form}(G)$ an *outer form* if $H^* \neq G$. Equivalently, H is an outer form if its image in $H^1(F, \mathrm{Out}(G))$ is non-trivial.

The last useful lemma we record is the following, which is easy (it follows from the proof of Proposition IV.4.6) and is left to the reader:

Lemma IV.4.7. *Let H be an element of $\mathsf{Form}(G)$ and H_0 an element of $\mathsf{QS}(G)$. Then, $H^* = H_0$ if and only if $\mathrm{cl}(H) = \mathrm{cl}(H_0)$.*

IV.4.2 Unitary groups: basic definitions and properties

We now specialize and elaborate the discussion from the previous subsection in the case when $G = \mathrm{GL}_{n,F}$. In particular, we recall the theory of *unitary groups* over F by which we mean forms of $\mathrm{GL}_{n,F}$. For simplicity we assume that F has characteristic 0.

To begin, let us fix the pair (B, T) in the case of $\mathrm{GL}_{n,F}$ to be the standard Borel B_n of upper triangular matrices, and the standard torus T_n of diagonal matrices. It is then not hard to check that the automorphisms of the associated based root datum are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. From this we deduce that we have natural bijections

$$\begin{aligned} H^1(F, \mathrm{Out}(\mathrm{GL}_{n,F})) &\cong \mathrm{Hom}_{\mathrm{cont.}}(\Gamma_F, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \{\text{étale algebras of degree 2 over } F\} \end{aligned} \tag{304}$$

which are identifications we freely make. Here an étale algebra of degree 2 over F means either $F \times F$, the *split étale algebra*, or a degree 2 extension E over F.

Before we continue, it will be helpful to clarify some notation concerning central simple algebras (or their generalizations *Azumaya algebras*) and their involutions. We begin by recalling the following definition.

Definition IV.4.8. *Let R be a (commutative unital) ring. Then an* Azumaya algebra *over R is a (possibly non-commutative) unital R-algebra A such that there exists some faithfully flat (commutative unital) R-algebra R' such that $A_{R'}$ is isomorphic to $\mathrm{Mat}_n(R')$ as an R'-algebra.*

We will only be interested in dealing with Azumaya algebras over degree 2 étale algebras over F, in which case such objects take a particularly simple form.

Namely, we have the following easy lemma:

Lemma IV.4.9. *Let R be a (commutative unital) ring.*

1. *If $R \to S$ is a ring map, and A is an Azumaya algebra over R, then A_S is an Azumaya algebra over S.*

2. *If R is a field, then an R-algebra A is an Azumaya algebra if and only if it's a central simple R-algebra.*

3. *If $R = F \times F$, where F is a field, then an R-algebra A is an Azumaya algebra if and only if $A \cong \Delta_1 \times \Delta_2$ where Δ_1 and Δ_2 are central simple F-algebras.*

Azumaya algebras can support involutions of particular interest to us, ones of the so-called *second kind*. We record here the rigourous definition:

Definition IV.4.10. *Let F be a field of characteristic 0 and E a degree 2 étale algebra over F and let us write σ for the non-trivial element of* $\mathrm{Gal}(E/F)$. *If A is an Azumaya algebra over E, then an* involution of the second kind *is a morphism $A \to A$, denoted $x \mapsto x^*$, satisfying the following properties:*

1. $(x+y)^* = x^* + y^*$ *for all* $x, y \in A$.

2. $(xy)^* = y^* x^*$ *for all* $x, y \in A$.

3. $x^* = \sigma(x)$ *for all* $x \in E$.

We shall often write $(A, *)$ for a pair of an Azumaya algebra and an involution of the second kind. To such a pair $(A, *)$ we can associate a *unitary group*:

Definition IV.4.11. *Let F be a field of characteristic 0 and E a 2-dimensional étale algebra over F. Then, for a pair $(A, *)$ of an an Azumaya algebra A over E and $*$ is an involution of the second kind we define the* unitary group of $(A, *)$, *denoted $U(A, *)$, to be the algebraic F-group whose R-points are given by*

$$U(A, *)(R) := \{x \in A_R : xx^* = 1\} \tag{305}$$

Let us now make the following elementary observation

Lemma IV.4.12. *Let F be a field of characteristic 0 and $E = F \times F$. Then, up to isomorphism, the only Azumaya algebras over E with an involution of the second kind are those of the form $(\Delta \times \Delta^{\mathrm{op}}, *_{switch})$ where Δ is a central simple F-algebra and*

$$*_{\mathrm{switch}}\,(x, y) = (y, x) \tag{306}$$

Moreover,

$$U(\Delta \times \Delta, *_{\mathrm{switch}}) \cong \Delta^\times \tag{307}$$

as algebraic groups over F.

Proof. The first claim is [KMRT98, Proposition 2.14]. The second claim is then clear. □

From this, we immediately deduce the following:

Lemma IV.4.13. *Let F be a field of characteristic 0 and let E be a degree 2 extension of F. Let $(\Delta, *)$ be a central simple E-algebra and let $U(\Delta, *)$ be its associated unitary group. Then, $U(\Delta, *)_E \cong \Delta^\times$.*

Proof. It's not hard to see that

$$U(\Delta, *)_E \cong U(\Delta_E, *_E) \tag{308}$$

where Δ_E is now an Azumaya algebra over $E \otimes_F E = E \times E$. By the previous lemma we know that

$$(\Delta_E, *_E) \cong (\Delta' \times \Delta', *_{\text{switch}}) \tag{309}$$

for some central simple E-algebra Δ'. Since Δ naturally embeds into Δ_E it's not hard to see that $\Delta' \cong \Delta$ and thus $U(\Delta, *)_E \cong \Delta^\times$ from the previous lemma. □

The last definition we require before returning to our analysis of the forms of $\mathrm{GL}_{n,F}$ is the follwing:

Definition IV.4.14. *Let F be a field of characteristic 0 and E a 2-dimensional étale algebra over F. A* Hermitian space relative to E/F *is a pair $(V, \langle -,- \rangle)$ consisting of a free E-module V together a non-degenerate F-linear pairing*

$$\langle -,- \rangle : V \times V \to E \tag{310}$$

such that $\langle -,- \rangle$ is E-linear in the first entry and satisfies

$$\langle v, w \rangle = \sigma(\langle w, v \rangle) \tag{311}$$

where σ is the non-trivial element of $\mathrm{Gal}(E/F)$.

For a Hermitian space $(V, \langle -,- \rangle)$ we define $U(V, \langle -,- \rangle)$ to be the algebraic F-group so that on F-algebras R we have the following:

$$U(V, \langle -,- \rangle)(R) := \{g \in \mathrm{GL}_R(V_R) : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V_E\} \tag{312}$$

Now, combining (304) with Proposition IV.4.5 we see that we have a bijection

$$\mathsf{QS}(\mathrm{GL}_n) \cong \{\text{étale algebras of degree 2 over } F\} \tag{313}$$

For an étale algebra E over F of degree 2 let us denote by $U_{E/F}(n)^*$ the element of $\mathsf{QS}(\mathrm{GL}_n)$ corresponding to E. We then have the following description of $U^*_{E/F}(n)$ which is well-known, and whose proof is elementary and left to the reader:

Lemma IV.4.15. *Let E be an etale algebra of degree 2 over F. If E is split then $U_{E/F}(n)^* \cong \mathrm{GL}_n$. If E is a degree 2 extension of F then there is an isomorphism*

$$U_{E/F}(n)^* \cong U(E^n, \langle -,- \rangle_0) \tag{314}$$

where

$$\langle x, y \rangle_0 := \overline{x}^\top J_N y \tag{315}$$

where

$$J_N = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \iddots & \iddots & 0 & 0 \\ 0 & \cdots & \vdots & 0 & 0 \\ (-1)^{N-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{316}$$

Thus, combining this lemma with Proposition IV.4.6 we deduce that

$$\mathsf{Form}(\mathrm{GL}_{n,F}) = \bigsqcup_E \mathsf{InnForm}(U_{E/F}(n)^*) \tag{317}$$

and, in particular, the outer forms of GL_n are precisely the inner forms of some $U_{E/F}(n)^*$ where E is a degree 2 extension of F.

The last thing we would like to do is explicate the structure of the pointed set $\mathsf{InnForm}(U_{E/F}(n)^*)$. Namely, we would like to claim the following:

Lemma IV.4.16. *The elements of* $\mathsf{InnForm}(U_{E/F}(n)^*)$ *are precisely* $U(A,*)$ *where* A *is an Azumaya algebra over* E *of* F*-dimension* $2n^2$ *over* F*.*

Proof. Let us first note that the fact that every form of $\mathrm{GL}_{n,F}$ is of the form $U(A,*)$ for some Azumaya algebra over a degree 2 etale algebra over F is classical (e.g. see [PR94, §2.3.4]). The fact that $\mathsf{InnForm}(\mathrm{GL}_{n,F})$ is just $\Delta^\times$ for a central simple algebra over F is also well-known (see loc. cit.).

Let us now deal with the non-split case. Let us note that by Lemma IV.4.7 that an element $H = U(A,*)$ of $\mathsf{Form}(\mathrm{GL}_{n,F})$ is in $\mathsf{InnForm}(U_{E/F}(n)^*)$ if and only if $\mathrm{cl}(H) = \mathrm{cl}(U_{E/F}(n)^*) = E$. Moreover, by functoriality we know that $\mathrm{cl}(H_E) = \mathrm{cl}(H)_E$ and since E is the unique non-trivial element of $H^1(F,\mathbb{Z}/2\mathbb{Z})$ with trivial image in $H^1(E,\mathbb{Z}/2\mathbb{Z})$. Thus, we see that H is in $\mathsf{InnForm}(U_{E/F}(n)^*)$ if and only if $\mathrm{cl}(H_E)$ is trivial. But, this is equivalent to saying that H_E is in $\mathsf{InnForm}(\mathrm{GL}_{n,F})$ which, by the previous paragraph, shows that $H_E \cong \Delta^\times$ for some central simple algebra Δ over E. Note then that this implies that $Z(H)_E$ is split. But, if A is an Azumaya algebra over E' then one can easily show compute that $Z(H)$ is the unique 1-dimensional torus over F split over E'. Thus, $E = E'$ as desired. □

We end this section with the well-known classification of unitary groups over local fields. We begin with the classification over $\mathbb{R}$:

Lemma IV.4.17. *There is a natural decomposition*

$$\mathsf{Form}(\mathrm{GL}_{n,\mathbb{R}}) = \mathsf{InnForm}(\mathrm{GL}_{n,\mathbb{R}}) \sqcup \mathsf{InnForm}(U_{\mathbb{C}/\mathbb{R}}(n)^*) \tag{318}$$

Moreover, we have that

$$\mathsf{InnForm}(\mathrm{GL}_{n,\mathbb{R}}) = \begin{cases} \{\mathrm{GL}_{n,\mathbb{R}}\} & \text{if} \quad \text{n odd} \\ \{\mathrm{GL}_{n,\mathbb{R}}, \mathrm{GL}_{\frac{n}{2}}(\mathbb{H})\} & \text{if} \quad \text{n even} \end{cases} \tag{319}$$

where $\mathbb{H}$ is the Hamiltonian quaternions and

$$\mathsf{InnForm}(U_{\mathbb{C}/\mathbb{R}}(n)^*) = \{U(p,q) : 0 \leqslant p \leqslant q \leqslant n \text{ and } p+q=n\} \tag{320}$$

where $U(p,q) = U(\mathbb{R}^n, \langle -, - \rangle_{(p,q)}$ where

$$\langle (x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle_{(p,q)} := x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_ny_n \tag{321}$$

Proof. The claim concerning the inner forms of $\mathrm{GL}_{n,\mathbb{R}}$ follows immediately from the observation that $H^1(\mathbb{R}, \mathrm{PGL}_n)$ injects in to $\mathrm{Br}(\mathbb{R})[2n]$ and since $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ the claim follows quite easily.

The second claim follows from a computation of $H^1(\mathbb{R}, (U_{\mathbb{C}/\mathbb{R}}(n)^*)^{\mathrm{ad}})$. Let us note that $U(n) := U(0,n)$ is an inner form of $U_{\mathbb{C}/\mathbb{R}}(n)$, since it's not an inner form of $\mathrm{GL}_{n,\mathbb{R}}$, and thus it suffices to compute $H^1(\mathbb{R}, U(n)^{\mathrm{ad}})$. Note though that by [Bor14, Theorem 9] this is equal to $H^1(\mathbb{R},T)/W_T(\mathbb{R})$ where T is a fundamental torus (i.e. a maximal torus of minimal split rank) in $U(n)^{\mathrm{ad}}$. But, $U(n)^{\mathrm{ad}}$ is $\mathbb{R}$-anisotropic so we can take T to be any maximal torus, namely $T = U(1)^n/Z(U(n))$ (where $U(1)$ is the unique non-split torus over $\mathbb{R}$). But, as can be easily calculated $H^1(\mathbb{R}, U(1)) = \mathbb{Z}/2\mathbb{Z}$ and thus $H^1(\mathbb{R},T) = ((\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z}))$ where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally in $(\mathbb{Z}/2\mathbb{Z})^n$. But, as can be easily checked (and as holds for any elliptic maximal torus in an $\mathbb{R}$-group), the group scheme W_T is constant. Thus, $W_T(\mathbb{R}) = W_T(\mathbb{C}) = S_n$. It's easy to check that the S_n action on $H^1(\mathbb{R},T)$ is the obvious one and thus

$$\begin{aligned} H^1(\mathbb{R}, U(n)^{\mathrm{ad}}) &\cong ((\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z}))/S_n \\ &\cong \{(p,q) \in \mathbb{N}^2 : 0 \leqslant p \leqslant q \text{ and } p+q=n\} \end{aligned} \tag{322}$$

It's then easy to check that $U(p,q)$, which is an inner form of $(U_{\mathbb{C}/\mathbb{R}}(n)^*$ is sent to (p,q) under the natural map $\mathsf{InnForm}(U(n)^{\mathrm{ad}}) \to H^1(\mathbb{R}, U(n)^{\mathrm{ad}})$ from where the conclusion follows. ∎

We now state the analogous classification of unitary groups over p-adic local fields:

Lemma IV.4.18. *Let F be a p-adic local field. There is a natural decomposition*

$$\mathsf{Form}(\mathrm{GL}_{n,F}) = \mathsf{InnForm}(\mathrm{GL}_{n,F}) \sqcup \bigsqcup_E \mathsf{InnForm}(U_{E/F}(n)^*) \tag{323}$$

where E travels over the degree 2 extensions of F (of which there are only finitely many). Moreover,

$$\mathsf{InnForm}(\mathrm{GL}_{n,F})\{\mathrm{GL}_k(D_{\frac{i}{j}}) : (i,j) = 1 \textit{ and } jk = n\} \tag{324}$$

where $D_{\frac{i}{j}}$ is the division algebra over F of invariant $\frac{i}{j}$ and

$$\mathsf{InnForm}(U_{E/F}(n)^*) \cong \begin{cases} \{e\} & \text{if} \quad n \,\text{odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n \,\text{even} \end{cases} \tag{325}$$

Proof. The first claim follows quite easily from the fact (see [Mil, Chapter IV,§4]) that the inner forms of $\mathrm{GL}_{n,F}$ are of the form $\Delta^\times$ where Δ is a central simple F-algebra of dimension n^2 and that such division algebras are all of the form $\mathrm{Mat}_m(D_{\frac{i}{j}})$ where $D_{\frac{i}{j}}$ is the division algebra of invariant $\frac{i}{j}$ (in the sense loc. cit.).

The second claim follows, again, by explicitly computing the pointed set $H^1(F,(U_{E/F}(n)^*)^{\mathrm{ad}})$. Let us det $H := (U_{E/F}(n)^*)^{\mathrm{ad}}$. We use [Kot86b, Theorem 1.2] to equate this to the computation of $\pi_0(Z(\widehat{H})^{\Gamma_F})$. But, $Z(\widehat{H}) \cong \mathbb{Z}/n\mathbb{Z}$ and it's not hard to check that Γ_F acts through its quotient $\mathrm{Gal}(E/F)$ and the non-trivial element of $\mathrm{Gal}(E/F)$ acts by multiplication by -1. The conclusion easily follows. □

IV.4.3 Anisotropicity and unitary groups

In this subsection we list some natural conditions that guarantee anisotropicity (modulo center) of unitary groups as well as the existence of elliptic maximal tori.

We start with the following:

Lemma IV.4.19. *Let E be a degree 2 étale algebra over F and set G^* to be $U^*_{E/F}(n)$. Let us set then set $G := U(A,*)$ to be an inner form of G^* Then:*

1. *If $E \cong F \times F$ then G satisfies that G^{ad} is F-anisotropic if and only if $G \cong D^\times$ where $D^\times$ is an F-central division algebra over F.*
2. *If E is a degree 2 extension of F, then G satisfies that G^{ad} is F-ansiotropic if $G \cong U(D,*)$ where D is an E-central division algebra.*

Before we prove this, it's useful to first recall the following:

Lemma IV.4.20. *Let F be a field of characteristic 0 and let G be a connected reductive group over F. Then, G^{ad} is F-anisotropic if and only if $G(F)$ contais no non-trivial unipotent elements.*

Proof. This follows from the contents of [BT72, §8]. □

Lemma IV.4.19. Suppose first that $E \cong F\times F$ and that G^{ad} is F-anisotropic. Then, we know from (or rather via the proof of) Lemma IV.4.16 that $G \cong \Delta^\times$ for some F-central simple algebra Δ. Note then that by the Artin-Wedderburn theorem that $\Delta^\times \cong \mathrm{GL}_m(D)$ for some (necessarily unique)

F-central division algebra D. If $m > 1$ then $G(F) = \mathrm{GL}_m(D)$ contains $\mathrm{GL}_m(F)$ which implies that $G(F)$ contains a unipotent element which contradicts Lemma IV.4.20. Thus $m = 1$ and thus $G \cong D^\times$ as desired.

Conversely, if $G \cong D^\times$ then to show that G^{ad} is anisotropic it suffices, by Lemma IV.4.20, to show that $G(F) = D^\times$ contains no non-trivial unipotent elements. But, note that the natural left action of $D^\times$ on itself gives an embedding $\iota : G \hookrightarrow \mathrm{GL}_F(D)$ and so it suffices to show that the map $D^\times \hookrightarrow \mathrm{GL}_F(D)$ on F-points has no unipotent elements in the image. But, if $u \in D^\times$ were unipotent then that would mean that $(\iota(u) - I)^n = 0$ for some $n \geqslant 1$. Note though that ι arises from an algebra embedding $\iota : D \hookrightarrow \mathrm{End}_F(D)$ which allows us to rewrite this equation as $\iota\left((u-1)^n\right) = 0$. Since ι is injective this implies that $(u-1)^n = 0$ and since D is a division algebra this implies that $u = 1$ as desired.

Suppose now that E is a degree 2 extension of F and let $G \cong U(D, *)$ where D is an E-central division algebra. By Lemma IV.4.20 it suffices to show that $U(D, *)(F)$ contais no non-trivial unipotent elements. Note though that, by definition, $U(D, *)$ is contained in $\mathsf{Res}_{E/F} D^\times$. So,

$$U(D, *)(F) \subseteq \mathsf{Res}_{E/F} D^\times = D^\times \tag{326}$$

The same argument as in the last paragraph then shows that no non-trivial unipotent elements exist. $\square$

Remark IV.4.21. One cannot change (2) in Lemma IV.4.19 to an if and only if. Indeed, note that over $\mathbb{R}$, for example, $U(n) := U(0, n)$ is anisotropic but is of the form $U(\mathrm{Mat}_n(\mathbb{R}), *)$.

We now would like to explain when unitary groups over a local field F contain elliptic maximal tori. If F is a p-adic local field this is a non-question by Lemma IV.1.6. Suppose now that $F = \mathbb{R}$ we then have the following:

Lemma IV.4.22. *Let $n > 1$ be an integer. Then, a form G of $\mathrm{GL}_{n,\mathbb{R}}$ has an elliptic maximal torus if:*

1. *If $n = 2$ and G arbitrary.*

2. *If $n > 2$ and G is an outer form of $\mathrm{GL}_{n,\mathbb{R}}$.*

Proof. By the classification in IV.4.17 and [Kal16, Lemma 3.2.1] it suffices to analyze for which n do $\mathrm{GL}_{n,\mathbb{R}}$ and $U(n) = U(0, n)$ have elliptic maximal tori. In the former case since the elliptic maximal tori in $\mathrm{GL}_{n,F}$, for any field F, are of the form $\mathsf{Res}_{F'/F} \mathbb{G}_{m,E}$ where F' is a degree n extension of F' it's clear that elliptic maximal tori exist if and only if $n = 2$. For the latter case since $U(n)$ is always $\mathbb{R}$-anisotropic the answer is clearly that elliptic maximal tori exist for all n. The deisred conclusion follows. $\square$

IV.4.4 Local-to-global definitions of unitary groups

We now explain the methodology for the construction of global unitary groups from local ones. In other words, we discuss the question of whether or not there is a (unique) unitary group over a number field F whose base change to F_v (for all places v of F) is some pre-perscribed unitary group.

So, let us fix F to be a global field (assumed to be a number field for convenience). From the last section we know that to give a form of $\mathrm{GL}_{n,F}$ is the same as to give a class in $H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F}))$. Note then that for every place v of F we have the usual localization map

$$H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F})) \to H^1(F_v, \mathrm{Aut}(\mathrm{GL}_{n,F})) \tag{327}$$

We can then assemble these maps to give a map

$$\mathrm{loc} : H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F})) \to \prod_v H^1(F_v, \mathrm{Aut}(\mathrm{GL}_{n,F})) \tag{328}$$

To begin, we have the following well-known lemma:

Lemma IV.4.23. *The localizaton map (327) is injective.*

Proof. Note that the sequence (291) for $\mathrm{GL}_{n,F}$

$$1 \to \mathrm{PGL}_{n,F} \to \mathrm{Aut}(\mathrm{GL}_{n,F}) \to \mathbb{Z}/2\mathbb{Z} \to 1 \tag{329}$$

splits. Thus, it suffices to prove that the maps

$$H^1(F, \mathrm{PGL}_{n,F}) \to \prod_v H^1(F_v, \mathrm{PGL}_{n,F}) \tag{330}$$

and

$$H^1(F, \mathbb{Z}/2\mathbb{Z}) \to \prod_v H^1(F_v, \mathbb{Z}/2\mathbb{Z}) \tag{331}$$

are injective.

To see that the map in (330) is injective, note that via the sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_{n,F} \to 1 \tag{332}$$

we get a commutative diagram

$$\begin{tikzcd} H^1(F, \mathrm{PGL}_{n,F}) \arrow[r] \arrow[d] & \prod_v H^1(F_v, \mathrm{PGL}_{n,F}) \arrow[d] \\ H^2(F, \mathbb{G}_m) \arrow[r] & \prod_v H^2(F_v, \mathbb{G}_m) \end{tikzcd} \tag{333}$$

where all vertical maps are injective (using Hilbert's theorem 90 together with the theory of twists as in [Ser13, Part I, §5.7]). Thus it suffices to show that the map

$$H^2(F,\mathbb{G}_m)\to\prod_v H^2(F_v,\mathbb{G}_m) \tag{334}$$

is injective. But, there are is an obvious commutative diagram

$$\begin{tikzcd} \mathrm{Br}(F) \arrow[r] \arrow[d] & \prod_v \mathrm{Br}(F_v) \arrow[d] \\ H^2(F,\mathbb{G}_m) \arrow[r] & \prod_v H^2(F_v,\mathbb{G}_m) \end{tikzcd} \tag{335}$$

where the vertical maps are isomorphisms. Thus, it suffices to show that

$$\mathrm{Br}(F)\to\prod_v \mathrm{Br}(F_v) \tag{336}$$

is injective. This follows form the fundamental exact sequence of class field theory (e.g. take the limit of the map in [Mil97, Chapter VII, Corollary 4.3]).

The fact that the map

$$H^1(F,\mathbb{Z}/2\mathbb{Z})\to\prod_v H^1(F_v,\mathbb{Z}/2\mathbb{Z}) \tag{337}$$

is injective follows from basic algebraic number theory. Namely, Kummer theory implies that this is equivalent to the injectivity of the map

$$K^\times/(K^\times)^2\to\prod_v K_v^\times/(K_v^\times)^2 \tag{338}$$

which is simple to see (e.g. see [Mil97, Chapter VII, Theorem 1.1]). □

As a corollary of the above we obtain the following:

Corollary IV.4.24. *For any degree 2 étale algebra E over F the natural map*

$$\mathrm{loc}_E:\mathsf{InnForm}(U_{E/F}(n)^*)\to\prod_v \mathsf{InnForm}(U_{E_v/F_v}(n)^*) \tag{339}$$

is injective.

Here we are abusing notation by denoting $E\otimes_F F_v$ by E_v. Of course, since E is a degree 2 étale algebra over F, E_v is a degree 2 étale algebra over F_v.

We would now like to describe the explicit image of loc_E. In other words, we'd like to discuss when a collection of inner forms of $U_{E_v/F_v}(n)^*$ for all places v of F is the simultaneous base change of some inner form of $U_{E/F}(n)^*$.

To do this it will be helpful to construct a map

$$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{340}$$

This map is given as follows (where we are using Lemma IV.4.17 and Lemma IV.4.18 without mention):

1. Assume that E_v is a degree 2 extension of F_v. Then:
   (a) if F_v is a p-adic local field then the map

   $$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{341}$$

   is the unique injective homomorphism.

   (b) if $F_v \cong \mathbb{R}$ then the map

   $$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{342}$$

   is defined as follows:

   $$\epsilon_v(U(p,q)) = \begin{cases} 1 & \text{if} \quad n \text{ odd} \\ \left\lfloor \dfrac{p-q}{2} \right\rfloor \mod 2 & \text{if} \quad n \text{ even} \end{cases} \tag{343}$$

   (c) Assume that $E_v \cong F_v \times F_v$. Then:
      i. if F_v is a p-adic local field then

      $$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{344}$$

      is the quotient map by $2(\mathbb{Z}/n\mathbb{Z})$ after making the identification $\mathsf{InnForm}(U^*_{E_v/F_v}(n)) \cong \mathbb{Z}/n\mathbb{Z}$ as above.

      ii. if $F_v \cong \mathbb{R}$ then

      $$\mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{345}$$

      is the unique injective homomorphism

Of course, we have neglected to say what happens when $F_v \cong \mathbb{C}$ in all cases, but here there are no non-trivial inner forms and so ϵ_v is just the trivial map.

We can now explicitly state which collections of local unitary groups come from a global unitary group:

Proposition IV.4.25. *Let F be a number field and let E be a degree 2 étale algebra over F. Then, the image of the injective map*

$$\mathsf{InnForm}(U_{E/F}(n)^*) \to \prod_v \mathsf{InnForm}(U_{E_v/F_v}(n)^*) \tag{346}$$

is the set of all tuples $(U_v)_v$ in $\prod_v \mathsf{InnForm}(U^*_{E_v/F_v}(n))$ such that the following two conditions hold:

1. *$U_v \cong U_{E_v/F_v}(n)^*$ for almost all v.*
2. *The equality*

$$\sum_v \epsilon_v(U_v) = 0 \tag{347}$$

holds as an element of $\mathbb{Z}/2\mathbb{Z}$.

Proof. This is contained in the contents of [Clo91, §2]. □

Remark IV.4.26. Note that ϵ_v is trivial for all v when n is odd, and so we see that in this case the only obstruction to a tuple $(U_v)_v$ of inner forms of $U^*_{E_v/F_v}(n)$ being the simultaneous base change of some inner form of $U^*_{E/F}(n)$ is that $U_v \cong U^*_{E_v/F_v}(n)$ for almost all v.

IV.4.5 Unitary groups with no relevant global endoscopy

We now discuss sufficient conditions for a unitary group $\mathbf{U}$ over a number field F, such that $\mathbf{U}^{\mathrm{ad}}$ is F-anisotropic, to have no relevant global endoscopy as in §I.5.

We begin by observing the following:

Lemma IV.4.27. *Let F be a global field and let E be a quadratic extension of E. Let $\mathbf{U}$ be an element of $\mathsf{InnForm}(U^*_{E/F}(n))$. Then, if $\mathbf{U} \cong U(D,*)$ for D an E-central division algebra then U has no relevant elliptic endoscopy.*

Proof. We would like to apply Proposition I.5.3. To do this we need to show that $\mathbf{U}^{\mathrm{ad}}$ is F-anisotropic and that $\mathbf{U}$ satisfies the Hasse principle. The former condition is Lemma IV.4.19. The latter is contained in [PS92, §6.7].

Now, let T be a maximal torus in $\mathbf{U}$. Then, we need to show that the containment $Z(\widehat{\mathbf{U}}) \subseteq \widehat{T}^{\Gamma_F}$ is an equality or, equivalently, that $\widehat{T}^{\Gamma_F} \subseteq Z(\widehat{\mathbf{U}})$. Note though that evidently

$$\widehat{T}^{\Gamma_F} \subseteq \widehat{T}^{\Gamma_E} = \widehat{T_E}^{\Gamma_E} \tag{348}$$

Note though that, by assumption, T_E is a maximal torus of $\mathbf{U}_E \cong D^\times$. But, all maximal tori of $D^\times$ are induced, say they are equal to $\mathsf{Res}_{M/E}\mathbb{G}_{m,E}$ where M is a degree n extension of E. It is then clear to see that $\widehat{T_E}^{\Gamma_E} \subseteq Z(\widehat{\mathbf{U}})$ as desired. □

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