Geometric coverings of rigid spaces

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I

de Jong covering spaces:

The good, the bad, and the non-overconvergent

The definition

Definition (Berkovich '93, de Jong '95)

A morphism $Y \to X$ of rigid K-spaces is called a *de Jong covering*, if there exists an overconvergent open cover $\{U_i\}$ of X such that Y_{U_i} is *almost split* (i.e. is an object of $\mathbf{UF\acute{E}t}_{U_i}$) for all i.

- $\mathbf{F\acute{E}t}_X = \left\{ \begin{matrix} \text{Finite étale} \\ X\text{-spaces} \end{matrix} \right\}, \qquad \mathbf{UF\acute{E}t}_X = \left\{ \begin{matrix} \text{Disjoint unions of} \\ \text{objects of } \mathbf{F\acute{E}t}_X \end{matrix} \right\}$
- *overconvergent open* : open of X coming from $[X] = X^{\mathrm{Berk}}$ (a.k.a. *partially proper opens*, a.k.a. *wide open subsets*).

$\mathbf{Cov}_X^{\mathrm{oc}}$: the category of de Jong covering spaces of X.

$\mathbf{UCov}_X^{\mathrm{oc}}$: the category of disjoint union of de Jong covering spaces of X.

The good I

$\mathbf{Cov}_X^{\mathrm{oc}}$ contains:

- $\mathbf{UF\acute{E}t}_X$,
- the category of 'topological coverings' of X,
- Ramero's category of locally algebraic étale coverings,
- Andre–Lepage's category of tempered covering spaces.

Example (de Jong, J.K. Yu)

The Gross–Hopkins period map

$$\pi_{\mathrm{GH}} : \mathcal{M}^{\mathrm{LT}}_{\mathbf{C}_p} \to \mathbf{P}^{1,\mathrm{an}}_{\mathbf{C}_p}$$

is a de Jong covering.

The good II

Theorem (de Jong)

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then,

$$(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}}), \qquad F_{\overline{x}}(Y) = Y_{\overline{x}}$$

is a tame infinite Galois category. In particular, if we set

$$\pi_1^{\mathrm{oc}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}}),$$

then one obtains an equivalence of categories

$$F_{\overline{x}} : \mathbf{UCov}_X^{\mathrm{oc}} \xrightarrow{\sim} \pi_1^{\mathrm{oc}}(X, \overline{x})\text{-}\,\mathbf{Set}\,.$$

$\pi_1^{\mathrm{oc}}(X, \overline{x})$: the *de Jong fundamental group*.

The good III

Theorem (de Jong)

Let X be a rigid K-space and $\overline{x}$ a geometric point of X. Then, there is a $\mathbf{Q}_\ell$-linear tensor functor

$$\omega_{\overline{x}} : \mathbf{Loc}(X_{\text{ét}}, \mathbf{Q}_\ell) \to \mathbf{Rep}(\pi_1^{\text{oc}}(X, \overline{x}))$$

which is an equivalence of X is connected.

$\mathbf{Loc}(X_{\text{ét}}, \mathbf{Q}_\ell)$: $\mathbf{Q}_\ell$-local systems on X.

$\mathbf{Rep}(\pi_1^{\text{oc}}(X, \overline{x}))$: category of continuous $\mathbf{Q}_\ell$-representations of $\pi_1^{\text{oc}}(X, \overline{x})$.

The bad I

Property	de Jong covering space
closed under disjoint unions	no
closed under compositions	no
oc open local	yes
admissible local	???
p.p. étale local	yes
étale local	???

The bad II

Question 1 (de Jong)

Are de Jong coverings admissible local on the target?

Question 2 (de Jong)

Is the pair $(\mathbf{UCov}_X^{\mathrm{adm}}, F_{\overline{x}})$ a tame infinite Galois category?

Question 3

What about étale local on the target? What about $(\mathbf{UCov}_X^{\text{ét}}, F_{\overline{x}})$?

The non-overconvergent I

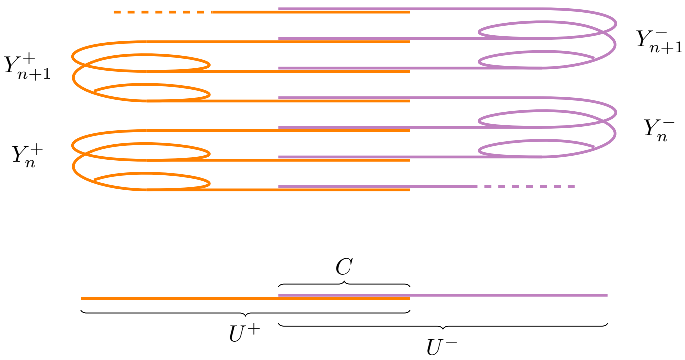
Theorem

Let K be a non-archimedean field of characteristic p, and let X be an annulus over K. Then, the containment $\mathbf{Cov}_X^{\mathrm{oc}} \subsetneq \mathbf{Cov}_X^{\mathrm{adm}}$ is strict.

- $X = \{|\varpi| \leq |x| \leq |\varpi|^{-1}\}$
- $U^- = \{|\varpi| \leq |x| \leq 1\}, \qquad U^+ = \{1 \leq |x| \leq |\varpi|^{-1}\}$
- $C = U^- \cap U^+ = \{|x| = 1\}$

Idea of construction: The covering $Y \to X$ is obtained by gluing two families $Y_n^{\pm}$ of Artin–Schreier coverings of $U^{\pm}$ which are split over shrinking overconvergent neighborhoods of C.

The non-overconvergent II



The non-overconvergent III

Question

Do there exist examples in mixed characteristic? We are confident the answer is yes, and one can adapt the example in equicharacteristic p.

Theorem

Let K be a discretely valued non-archimedean field of equicharacteristic 0. *Then, for any smooth (qpc+qs) X one has the equality*

$$\mathbf{Cov}_X^{\mathrm{oc}} = \mathbf{Cov}_X^{\mathrm{adm}} = \mathbf{Cov}_X^{\text{ét}}.$$

II

Geometric arcs and geometric coverings

de Jong's independence of basepoint result

Theorem (de Jong)

Let X be a connected rigid K-space and $\overline{x}$ and $\overline{y}$ geometric points of X. Then, $\pi_1^{\mathrm{oc}}(X;\overline{x},\overline{y})$ is non-empty.

$$\pi_1^{\mathcal{C}}(X;\overline{x},\overline{y}) = \mathrm{Isom}(F_{\overline{x}}|_{\mathcal{C}}, F_{\overline{y}}|_{\mathcal{C}}), \qquad \mathcal{C} \subseteq \mathbf{\acute{E}t}_X$$

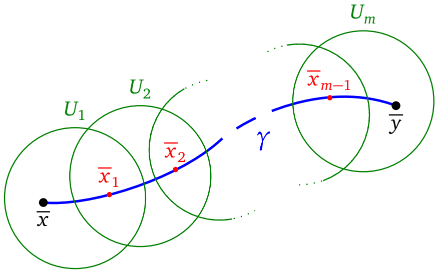
$$\pi_1^{\mathbf{UF\acute{E}t}_X}(X;\overline{x},\overline{y}) = \pi_1^{\mathrm{alg}}(X;\overline{x},\overline{y}), \quad \text{algebraic étale paths/group}$$

$$\pi_1^{\mathbf{Cov}^{\mathrm{oc}}}(X;\overline{x},\overline{y}) = \pi_1^{\mathrm{oc}}(X;\overline{x},\overline{y}).$$

Outline of de Jong's proof of independence of base point I

Step 1: Let γ be an arc connecting x and y in $[X]$.

Step 2: Define when an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $[X]$ is 'linearly arranged along γ'.



Set

$$\mathbf{Cov}_{\mathcal{U}} = \{Y \in \mathbf{Cov}_X : Y_U \in \mathbf{UF\acute{E}t}_U \text{ for all } U \in \mathcal{U}\}.$$

Outline of de Jong's proof of independence of base point II

Step 3: $\mathbf{Cov}_X^{\mathrm{oc}} = \varinjlim_{\mathcal{U}} \mathbf{Cov}_{\mathcal{U}}$, so $\pi_1^{\mathrm{oc}}(X;\overline{x},\overline{y}) = \varprojlim_{\mathcal{U}} \pi_1^{\mathcal{U}}(X;\overline{x},\overline{y})$.

Step 4: Define $K_{\mathcal{U}}$ to be image of the 'composition map'

$$\pi_1^{\mathrm{alg}}(U_1;\overline{x},\overline{x}_1) \times \cdots \times \pi_1^{\mathrm{alg}}(U_m;\overline{x}_{m-1},\overline{y}) \to \pi_1^{\mathcal{U}}(X;\overline{x},\overline{y})$$

and use the fact that $K_{\mathcal{U}}$ is compact to deduce $\varprojlim_{\mathcal{U}} K_{\mathcal{U}} \neq \varnothing$.

Geometric arc

Definition

A *geometric arc* $\overline{\gamma}$ in X consists of:

- an arc γ in $[X]$,
- for every point $z \in \gamma$, a geometric point $\overline{z}$ of X anchored at $z^{\max}$,
- for every subarc $[a,b] \subseteq \gamma$, and every open oc neighborhood U of $[a,b]$ an element $\iota_{a,b}^{U} \in \pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b})$,

such that:

1. for all $[a,b] \subseteq \gamma$ and open oc neighborhoods $U \subseteq U'$ of $[a,b]$
$$\pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b}) \to \pi_1^{\mathrm{alg}}(U';\overline{a},\overline{b})$$
maps $\iota_{a,b}^{U}$ to $\iota_{a,b}^{U'}$,
2. for subarcs $[a,b]$ and $[b,c]$ of γ, and for every $U \subseteq X$ open oc neighborhood of $[a,c] = [a,b] \cup [b,c]$, the composition map
$$\pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b}) \times \pi_1^{\mathrm{alg}}(U;\overline{b},\overline{c}) \to \pi_1^{\mathrm{alg}}(U;\overline{a},\overline{c})$$
maps $(\iota_{a,b}^{U}, \iota_{b,c}^{U})$ to $\iota_{a,c}^{U}$.

Geometric path connectedness

Theorem (de Jong, Berkovich, Achinger–Lara–Y.)

Suppose X is connected and let x and y be maximal points of X. Then, there exists an extension L/K, smooth connected affinoid L-curves C_i, and maps $C_i \to X$ such that

1. $\mathrm{im}(C_i \to X) \cap \mathrm{im}(C_{i+1} \to X)$ *is non-empty,*
2. $x \in \mathrm{im}(C_1 \to X)$, *and* $y \in \mathrm{im}(C_m \to X)$

Theorem

Let X be a connected, smooth, and separated rigid K-curve. Then, for any two maximal geometric points $\overline{x}$ and $\overline{y}$ of X there exists a geometric arc $\overline{\gamma}$ that has $\overline{x}, \overline{y}$ as its endpoints.

Morally: connected rigid K-spaces are 'geometric path connected'.

Geometric coverings

Definition

A map $Y \to X$ satisfies *unique lifting of geometric arcs* if for all geometric arcs $\overline{\gamma}$ of X with left geometric endpoint $\overline{x}$, and every lift $\overline{x}'$ of $\overline{x}$, there exists a unique lift $\overline{\gamma}'$ of $\overline{\gamma}$ with left geometric endpoint $\overline{x}'$.

Definition

A morphism $Y \to X$ of rigid K-spaces is called a *geometric covering* if it is

1. étale,
2. partially proper,
3. and for all test curves $C \to X$ the map $Y_C \to C$ satisfies unique lifting of geometric arcs.

test curve : a map $C \to X$ over K where C is a smooth separated rigid L-curve for some extension L/K.

Cov$_X$: category of geometric coverings of X.

Properties of geometric coverings

Property	de Jong covering space	geometric covering space
closed under disjoint unions	no	yes
closed under compositions	no	yes
oc open local	yes	yes
admissible local	no	yes
p.p. étale local	yes	yes
étale local	no	yes

$$\rightsquigarrow$$

$$\mathbf{Cov}_X^{\mathrm{oc}} \subseteq \mathbf{Cov}_X^{\mathrm{adm}} \subseteq \mathbf{Cov}_X^{\text{ét}} \subseteq \mathbf{Cov}_X$$

The geometric arc fundamental group

Theorem

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category. In particular, if we set

$$\pi_1^{\mathrm{ga}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

then we have an equivalence

$$F_{\overline{x}} : \mathbf{Cov}_X \xrightarrow{\sim} \pi_1^{\mathrm{ga}}(X, \overline{x})\text{-}\,\mathbf{Set}$$

$\pi_1^{\mathrm{ga}}(X, \overline{x})$: the *geometric arc fundamental group*.

NB: The non-emptiness of $\pi_1^{\mathrm{ga}}(X; \overline{x}, \overline{y})$ is now the easy part!

Answer to Question 2 and Question 3

Theorem

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, for $\tau \in \{\mathrm{adm}, \text{ét}\}$ the pair $(\mathbf{UCov}_X^{\tau}, F_{\overline{x}})$ is a tame infinite Galois category. In particular, if we set

$$\pi_1^{\tau}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}}),$$

then we get an equivalence

$$F_{\overline{x}} : \mathbf{UCov}_X^{\tau} \xrightarrow{\sim} \pi_1^{\tau}(X, \overline{x})\text{-}\,\mathbf{Set}\,.$$

We get a series of maps of topological groups with dense image

$$\pi^{\mathrm{ga}}(X, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\mathrm{adm}}(X, \overline{x}) \to \pi_1^{\mathrm{oc}}(X, \overline{x})$$

III

Relationship to Bhatt–Scholze's geometric coverings and AVC

Bhatt–Scholze's geometric coverings

Definition (Bhatt–Scholze)

Let X be a locally topologically Noetherian scheme. A morphism $Y \to X$ is a *geometric covering* if it's étale and partially proper.

$\mathbf{Cov}_X$: the category of geometric coverings of X.

$\pi_1^{\text{proét}}(X, \overline{x})$: the fundamental group of the tame infinite Galois category $(\mathbf{Cov}_X, F_{\overline{x}})$ (Bhatt–Scholze).

Examples

Example of geometric covering

Let X be the projective nodal cubic. Then, there is a natural étale map $Y \to \mathbb{P}^1_k$, where Y is the bi-infinite chain Y of $\mathbb{P}^1_k$'s, is a geometric covering.

Example of weirdness in rigid geometry

Let $\mathbf{D}_K$ be the closed unit disk, and $\mathbf{D}^\circ_K$ the open unit disk. Then, $\mathbf{D}^\circ_K \hookrightarrow \mathbf{D}_K$ is étale and partially proper.

AVC

Definition

A map $Y \to X$ of rigid K-spaces *satisfies the arcwise valuative criterion (AVC)* if for every commutative square of solid arrows

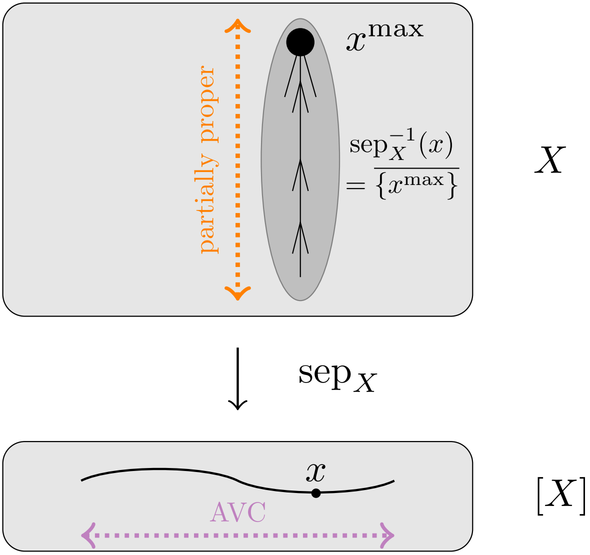
$$\begin{array}{ccc} [0,1) & \longrightarrow & [Y] \\ \big\downarrow & \nearrow & \big\downarrow {\scriptstyle [f]} \\ [0,1] & \xrightarrow[i]{} & [X] \end{array}$$

where i is a topological embedding, there exists a unique dotted arrow making the diagram commute.

Example

$\mathbf{D}_K^\circ \hookrightarrow \mathbf{D}_K$ does not satisfy AVC.

A picture of AVC and partial properness



AVC and geometric coverings

Theorem

Let C be a smooth separated rigid K-curve. Then, for an étale and partially proper map $Y \to C$, the following are equivalent:

1. *$Y \to C$ satisfies unique lifting of geometric arcs,*
2. *$Y \to C$ satisfies AVC.*

Morally: A geometric covering is a map of rigid spaces which:

1. is étale,
2. satisfies a geometric valuative criterion (partial properness),
3. satisfies a topological valuative criterion (AVC).

Generic fibers of geometric coverings

Theorem

Let $\mathfrak{Y} \to \mathfrak{X}$ be an étale morphism of admissible formal $\mathcal{O}_K$-scheme, with $\mathfrak{X}$ qpc. Then, the following are equivalent:

1. $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ *is a de Jong covering,*
2. $\mathfrak{Y}_k \to \mathfrak{X}_k$ *is a geometric covering,*
3. $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ *is partially proper.*

$$\mathbf{Cov}_{\mathfrak{X}_k} \overset{(*)}{\hookrightarrow} \mathbf{Cov}^{\mathrm{oc}}_{\mathfrak{X}_\eta} \subseteq \mathbf{Cov}_{\mathfrak{X}_\eta}$$

Example

Let $\mathfrak{X}$ be the natural deformation of the projective nodal cubic X, and let $\mathfrak{Y} \to \mathfrak{X}$ be the unique étale deformation of $Y \to X$. Then, $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a de Jong covering. In fact, it's the Tate uniformization map.

IV

Relationship to local systems in the pro-étale topology

The pro-étale site

Let X be a (locally Noetherian) adic space. In *p-adic Hodge theory for rigid-analytic varieties* Scholze defined a site $X_{\text{proét}}$ with

- underlying category a certain subcategory of $\mathbf{Pro}(\mathbf{\acute{E}t}_X)$,
- covers being certain jointly surjective sets of maps satisfying some pro-presentation properties.

$\mathbf{Loc}(X_{\text{proét}})$: locally constant sheaves of sets on $X_{\text{proét}}$.

$h^{\sharp}_{\mathsf{Y},\text{proét}}$: sheafification of the representable presheaf associated to Y.

Relationship between $\mathbf{Cov}_X^{\text{ét}}$ and $\mathbf{Loc}(X_{\text{proét}})$

Theorem

Let X be an analytic locally Noetherian adic space. Then, the functor

$$\mathbf{Cov}_X^{\text{ét}} \to \mathbf{Sh}(X_{\text{proét}}), \qquad Y \mapsto h^{\sharp}_{Y,\text{proét}}$$

is fully faithful with essential image $\mathbf{Loc}(X_{\text{proét}})$.

Corollary

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then,

$$\mathbf{ULoc}(X_{\text{proét}}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})\text{-}\,\mathbf{Set}, \qquad \mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$$

is an equivalence.

V

Final thoughts

Further Questions

- Are the containments $\mathbf{Cov}_X^{\mathrm{adm}} \subseteq \mathbf{Cov}_X^{\text{ét}} \subseteq \mathbf{Cov}_X$ strict?
- Is there a theory of 'tame arcs' that would allow you to avoid the introduction of curves?
- Is there a topology τ for which $\mathbf{Cov}_X = \mathbf{Loc}(X_\tau)$?
- Can geometric arcs be understood in terms of morphisms of topoi $\mathbf{Sh}([0,1]) \to \mathbf{Sh}(X_{\text{p.p.,ét}})$? (suggested by Scholze)
- Can geometric arcs be used to study other things (e.g. exit paths and constructible sheaves)?

Thanks for listening!