An Approach to the Characterization of the Local Langlands Correspondence

Alexander Bertoloni Meli[1] (Corresponding author)
Alex Youcis[2]

[1]University of Michigan, 530 Church St, Ann Arbor, MI 48109, United States
email: abertolo@umich.edu

[2]IMPAN, Jana i Jędrzeja Śniadeckich 8, 00-656 Warsaw, Poland
email: ayoucis@impan.pl

October 19, 2020

Abstract

In this paper, we explain how Scholze's characterization of the local Langlands correspondence for general linear groups can be made to work for a more general class of reductive groups. Our paper shows, in the case of supercuspidal L-parameters, how to overcome the key representation-theoretic problem of understanding how Scholze's construction works in a setting with non-singleton L-packets.

1 Introduction

In [Sch13b], Scholze gave a new construction of the local Langlands correspondence for $\mathrm{GL}_{n,F}(F)$, for F a p-adic field. A key component of Scholze's paper is that he was able to characterize his correspondence by an explicit equation relating his construction to certain functions $f_{\tau,h}$ of geometric provenance.

A major appeal of Scholze's characterization of the local Langlands correspondence for $\mathrm{GL}_{n,F}(F)$ is that it should be possible to generalize to the setting of a general reductive group G. In contrast, the standard characterization for the $\mathrm{GL}_{n,F}(F)$ case, as first described fully in [Hen00], is specialized to work for $\mathrm{GL}_{n,F}(F)$. Similarly, the characterizations for classical groups following from [Art13] use twisted endoscopy to reduce to the case of $\mathrm{GL}_{n,F}(F)$ where Henniart's characterization can be applied. Unfortunately, many groups can not be related to $\mathrm{GL}_{n,F}(F)$ via endoscopy and for these cases one needs a more general approach. Moreover, even in cases where this is possible, having a characterization internal to the group G is desirable.

The two major complications of generalizing the results of [Sch13b] to arbitrary groups G are:

(Q1) How to generalize the functions $f_{\tau,h}$ of [Sch13b] and prove they satisfy analogous equations.

(Q2) Decide whether two constructions of the local Langlands correspondence satisfying the generalized equations must coincide.

The question in **(Q1)** has been considered by several authors. Namely, the functions $f_{\tau,h}$ were generalized in [Sch13a] to PEL/EL type cases (and in [You19] to abelian type cases) and in [SS13], Scholze and Shin give a precise conjecture generalizing the main equation in [Sch13b]. We call the generalized equation in [SS13] the *Scholze–Shin equation*. In [SS13] it is proven that these equations hold in EL type cases and in [BMY19], the authors prove the Scholze–Shin equations hold for supercuspidal parameters of unramified unitary groups (with the local Langlands conjecture as in [Mok15]).

That said, the question in **(Q2)** does not seem to have been studied at all in [SS13] or subsequent work. This is surprising since while **(Q2)** is nearly trivial in the case of $\mathrm{GL}_{n,F}(F)$, it becomes highly non-trivial for more general groups due to the existence of non-singleton L-packets. It is important to note that **(Q2)** is of a purely representation-theoretic nature and hence is non-obvious even for groups G which have well-known associated geometric objects. For instance, in the case where G is a quasi-split unitary group **(Q2)** is non-obvious despite the fact that Shimura varieties for globalizations of these groups are well-understood.

The goal of this paper is to resolve **(Q2)**, modulo some extra conditions, for a substantial class of groups in the case of *supercuspidal L-parameters*. These are the L-parameters $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ that do not factor through a proper Levi subgroup ${}^L M \subset {}^L G$ and where the $\mathrm{SL}_2(\mathbb{C})$-factor acts trivially. The supercuspidal L-parameters are conjecturally those whose L-packet consists entirely of supercuspidal representations. This is a natural class of parameters to consider since their conjectured properties are well understood and these are precisely the parameters that appear in Kaletha's type-theoretic construction of the local Langlands correspondence [Kal19]. We call the assignment of packets to supercuspidal L-parameters satisfying some basic properties a *supercuspidal local Langlands correspondence* (see §3 for a precise definition).

Our main result is as follows:

Theorem 1.1 (Imprecise version of Theorem 3.3)**.** *Let G be a 'good' reductive group over F and suppose Π^i for $i = 1, 2$ are supercuspidal local Langlands correspondences for G that are compatible with elliptic endoscopy, satisfy standard desiderata, and such that*

1. *Each representation appearing in a singleton L-packet of Π^1 is contained in an L-packet for Π^2,*

2. *The L-packets of Π^1 and Π^2 satisfy the Scholze–Shin equations with respect to the same set of functions $\{f^{\mu}_{\tau,h}\}$.*

Then $\Pi^1 = \Pi^2$.

Remark 1.2. The reader will note that to make precise the compatibility with elliptic endoscopy in the above statement, one requires constructions of Π^i for elliptic endoscopic groups of G. These groups will typically be quite similar to G and so we have ignored this for the purposes of the introduction.

Remark 1.3. Our method, as currently stated, cannot hope to handle all groups G. The definition of a 'good' group is somewhat technical and discussed in §3. Forms of $\mathrm{GL}_{n,F}$ and $\mathrm{SO}_{2n+1,F}$ are 'good' while forms of $\mathrm{Sp}_{2n,F}$ and $\mathrm{SO}_{2n,F}$ are not in general.

Condition 1. in Theorem 1.1 seems unavoidable since we only consider supercuspidal parameters and there is not at present a precise expectation as to which subset of the set of irreducible $G(F)$-representations these should correspond to. We expect that most constructions of a supercuspidal local Langlands correspondence will identify the relevant set and hence we have some hope that this condition can be checked on a case by case basis. For instance, the union of the L-packets appearing in Kaletha's construction in [Kal19] is precisely the set of *non-singular* supercuspidals for the groups he considers.

The idea for the proof of Theorem 1.1 is simple, but illuminating, and is inspired by ideas present in [Art13]. The four major steps are as follows:

Step 1: Every parameter ψ of G with non-singleton L-packet factorizes through a parameter ψ_H of an elliptic hyperendoscopic group H of G with the property that ψ_H has a singleton L-packet.

Step 2: Using the endoscopic character identities and **Step 1** the characterization is reduced to the case of singleton packets.

Step 3: Show our desiderata imply the atomic stability property for supercuspidal L-packets. This property states that for the representations we consider, a stable linear combination of distribution characters of irreducible $G(F)$-representations is in fact a linear combination of the stable characters attached to each L-packet.

Step 4: The Scholze–Shin equations plus atomic stability are enough to pin down singleton packets by the Brauer–Nesbitt theorem.

Note that the Scholze–Shin equations via step **Step 4** allow us to avoid the use of twisted endoscopy.

Combining the above theorem with the aforementioned proof that the Scholze–Shin equations hold for the 'supercuspidal' local Langlands conjecture as in [Mok15], we obtain:

Theorem 1.4 (See Theorem 6.2)**.** *The supercuspidal local Langlands correspondence for unramified unitary groups as given by* [Mok15] *is characterized by* Theorem 1.1.

Acknowledgements

The authors are greatly indebted to Kaoru Hiraga for describing the argument for Proposition 4.2. This proposition allowed the authors to remove 'atomic stability for singletons' as an assumption in Theorem 3.3.

The authors would like to kindly thank Tasho Kaletha, Sug Woo Shin, and Masao Oi for very helpful conversations concerning this paper. Finally, the authors would like to thank Mr. and Mrs. Bertoloni Meli for providing an excellent environment to write a paper replete with apples and rapini.

During the completion of this work, the first author was partially funded by NSF RTG grant 1646385. The second author was partially supported by the funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 802787).

2 Notation

The following notation will be used throughout the rest of the paper unless stated otherwise.

Let F be a p-adic local field. Fix an algebraic closure $\overline{F}$ and let F^{un} be the maximal unramified extension of F in $\overline{F}$. Let L be the completion of F^{un} and fix an algebraic closure $\overline{L}$.

Let G be a (connected) reductive group over F. We denote by $G(F)^{\mathrm{reg}}$ the regular semisimple elements in $G(F)$ and by $G(F)^{\mathrm{ell}}$ the subset of elliptic regular semisimple elements. We denote by D, or D_G, the Harish-Chandra discriminant map on $G(F)$. If $\gamma, \gamma' \in G(F)$ are stably conjugate we denote this by $\gamma \sim_{\mathrm{st}} \gamma'$.

Let $\widehat{G}$ be the connected Langlands dual group of G and let ${}^LG$ be the Weil group version of the L-group of G as defined in [Kot84b, §1]. We denote the set of irreducible smooth representations of $G(F)$ by $\mathrm{Irr}(G(F))$ and by $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ the subset of supercuspidal representations. For a finite group C the notation $\mathrm{Irr}(C)$ means all irreducible $\mathbb{C}$-valued representations of C.

A *supercuspidal Langlands parameter* is an L-parameter (see [Bor79, §8.2]) $\psi : W_F \to {}^LG$ such that the image of ψ is not contained in a proper

Levi subgroup of ${}^LG$. We say that supercuspidal parameters ψ and ψ' are equivalent if they are conjugate in $\widehat{G}$ and denote this by $\psi \sim \psi'$. Let C_ψ be the centralizer of $\psi(W_F)$ in $\widehat{G}$. Then by [Kot84b, §10.3.1], ψ is supercuspidal if and only if the identity component C_ψ° of C_ψ is contained in $Z(\widehat{G})^{\Gamma_F}$. We define the group $\overline{C_\psi} := C_\psi/Z(\widehat{G})^{\Gamma_F}$ which is finite by our assumptions on ψ. For the sake of comparison, in [Kal16a, Conj. F], Kaletha defines $S_\psi^\natural := C_\psi/(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ$. For ψ a supercuspidal parameter, we have

$$S_\psi^\natural = C_\psi. \tag{1}$$

Indeed,

$$(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ = (C_\psi^\circ \cap [\widehat{G}]_{\mathrm{der}})^\circ \subset (Z(\widehat{G})^{\Gamma_F} \cap [\widehat{G}]_{\mathrm{der}})^\circ = \{1\}, \tag{2}$$

from where the equality follows.

Define $Z^1(W_F, G(\overline{L}))$ to be the set of continuous cocycles of W_F valued in $G(\overline{L})$ and let $\mathbf{B}(G) := H^1(W_F, G(\overline{L}))$ be the corresponding cohomology group. Let $\kappa : \mathbf{B}(G) \to X^*(Z(\widehat{G})^{\Gamma_F})$ be the Kottwitz map as in [Kot97].

An *elliptic endoscopic datum* of G (cf. [Kot84b, 7.3-7.4]) is a triple (H, s, η) of a quasisplit reductive group H, an element $s \in Z(\widehat{H})^{\Gamma_F}$, and a homomorphism $\eta : \widehat{H} \to \widehat{G}$. We require that η gives an isomorphism

$$\eta : \widehat{H} \to Z_{\widehat{G}}(\eta(s))^\circ, \tag{3}$$

that the $\widehat{G}$-conjugacy class of η is stable under the action of Γ_F, and that $(Z(\widehat{H})^{\Gamma_F})^\circ \subset Z(\widehat{G})$.

An *extended elliptic endoscopic datum* of G is a triple $(H, s, {}^L\eta)$ such that ${}^L\eta : {}^LH \to {}^LG$ and $(H, s, {}^L\eta|_{\widehat{H}})$ gives an elliptic endoscopic datum of G.

An *extended elliptic hyperendoscopic datum* is a sequence of tuples of data $(H_1, s_1, {}^L\eta_1), \ldots, (H_k, s_k, {}^L\eta_k)$ such that $(H_1, s_1, {}^L\eta_1)$ is an extended elliptic endoscopic datum of G, and for $i > 1$, the tuple $(H_i, s_i, {}^L\eta_i)$ is an extended elliptic endoscopic datum of H_{i-1}. An *elliptic hyperendoscopic group* of G is a quasisplit connected reductive group H_k appearing in an extended elliptic hyperendoscopic datum for G as above.

3 Statement of main result

Throughout the rest of the paper we assume that our groups G satisfy the following assumption:

(Ext) For each elliptic hyperendoscopic group H of G and each elliptic endoscopic datum (H', s, η') of H, one can extend (H', s, η') to an extended elliptic endoscopic datum $(H', s, {}^L\eta')$ such that ${}^L\eta' : {}^LH' \to {}^LH$.

Remark 3.1. The authors are not aware of any example for G a group over F where this property does not hold. If G and all its hyperendoscopic groups have simply connected derived subgroup, then **(Ext)** follows from [Lan79, Prop. 1]. In particular, unitary groups satisfy **(Ext)**.

All elliptic endoscopic data (H, s, η) for G a symplectic or special orthogonal group can also be extended to a datum $(H, s, {}^L\eta)$ ([Kal16a, pg.5]). Since the elliptic endoscopic groups of symplectic and special orthogonal groups are products of groups of this type ([Wal10, §1.8]), it follows that symplectic and special orthogonal groups also satisfy **(Ext)**.

One could likely remove the assumption **(Ext)** altogether at the cost of having to consider z-extensions of endoscopic groups (see [KS99]) and perhaps slightly modify the statement of Theorem 3.3 to account for these extra groups.

We now state the main result. Let us fix G^* to be a quasi-split reductive group over F. We define a *supercuspidal local Langlands correspondence* for a group G^* to be an assignment

$$\Pi_H : \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{Supercuspidal } L\text{-parameters} \\ \text{for } H \end{array} \right\} \to \left\{ \begin{array}{c} \text{Subsets of} \\ \mathrm{Irr}^{\mathrm{sc}}(H(F)) \end{array} \right\}, \tag{4}$$

for every elliptic hyperendoscopic group H of G^* satisfying the following properties:

(Dis) If $\Pi_H(\psi) \cap \Pi_H(\psi') \neq \emptyset$ then $\psi \sim \psi'$.

(Bij) For each Whittaker datum $\mathfrak{w}_H$ of H, a bijection

$$\iota_{\mathfrak{w}_H} : \Pi_H(\psi) \to \mathrm{Irr}(\overline{C_\psi}). \tag{5}$$

This bijection $\iota_{\mathfrak{w}_H}$ gives rise to a pairing

$$\langle -, - \rangle_{\mathfrak{w}_H} : \Pi_H(\psi) \times \overline{C_\psi} \to \mathbb{C}, \tag{6}$$

defined as follows:

$$\langle \pi, s \rangle_{\mathfrak{w}_H} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_H}(\pi)). \tag{7}$$

(St) For all supercuspidal L-parameters ψ of H, the distribution

$$S\Theta_\psi := \sum_{\pi \in \Pi_H(\psi)} \langle \pi, 1 \rangle \Theta_\pi, \tag{8}$$

is stable and does not depend on the choice of $\mathfrak{w}_H$.

(ECI) For all extended elliptic endoscopic data $(H', s, {}^L\eta)$ for H and all $h \in \mathscr{H}(H(F))$, suppose ψ^H is a supercuspidal L-parameter of H that factors through ${}^L\eta$ by some parameter $\psi^{H'}$. Then such a $\psi^{H'}$ must be supercuspidal and we assume it satisfies the *endoscopic character identity*:

$$S\Theta_{\psi^{H'}}(h^{H'}) = \Theta^s_{\psi^H}(h), \tag{9}$$

where we define $h^{H'}$ to be a transfer of h to H' (e.g. see [Kal16a, §1.3]) and we define

$$\Theta^s_{\psi^H} := \sum_{\pi \in \Pi_H(\psi^H)} \langle \pi, s \rangle \Theta_\pi. \tag{10}$$

the s-twisted character of ψ^H.

Suppose now that $z_{\mathrm{iso}} \in Z^1(W_F, G(\overline{L}))$ projecting to an element of $\mathbf{B}(G)_{\mathrm{bas}}$. Let G be the inner form of G^* corresponding to the projection of z_{iso} to $Z^1(W_F, \mathrm{Aut}(G)(\overline{F}))$. We then define a *supercuspidal local Langlands correspondence* for the *extended pure inner twist* (G, z_{iso}) (cf. [Kal16a, §2.5]) to be a supercuspidal local Langlands correspondence for G^* as well as a correspondence

$$\Pi_{(G,z_{\mathrm{iso}})} : \left\{ \begin{matrix} \text{Supercuspidal } L\text{-parameters} \\ \text{for } G \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Subsets of} \\ \mathrm{Irr}^{\mathrm{sc}}(G(F)) \end{matrix} \right\}, \tag{11}$$

satisfying

(Bij') For each Whittaker datum $\mathfrak{w}_G$ of G, a bijection

$$\iota_{\mathfrak{w}_G} : \Pi_G(\psi) \to \mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}}), \tag{12}$$

where $\mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}})$ denotes the set of equivalence classes of irreducible algebraic representations of C_ψ with central character on $Z(\widehat{G})^{\Gamma_F}$ equal to $\chi_{z_{\mathrm{iso}}} := \kappa(\overline{z_{\mathrm{iso}}})$. This gives rise to a pairing

$$\langle -, - \rangle_{\mathfrak{w}_G} : C_\psi \times \mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}}) \to \mathbb{C}, \tag{13}$$

defined as

$$\langle \pi, s \rangle_{\mathfrak{w}_G} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_G}(\pi)). \tag{14}$$

(ECI') For all supercuspidal parameters ψ of G and all extended elliptic endoscopic data $(H, s, {}^L\eta)$ of G such that ψ factors as $\psi = {}^L\eta \circ \psi^H$, there is an equality

$$\Theta^s_{\psi^H}(h^H) = S\Theta_\psi(h), \tag{15}$$

where $h \in \mathscr{H}(G(F))$ and $S\Theta_\psi$ is independent of choice of Whittaker datum in **(Bij')**.

For a supercuspidal local Langlands correspondence Π for (G, z_{iso}) we say that a subset of $\mathrm{Irr}(H(F))$ of the form $\Pi_\psi(H)$ is a *supercuspidal L-packet* for Π_H. We furthermore say that an element π of $\mathrm{Irr}(H(F))$ is *Π_H-accessible* if π is in a supercuspidal L-packet for Π_H. We say that an element π of $\mathrm{Irr}(H(F))$ is *singly Π_H-accessible* if $\{\pi\}$ is an L-packet for Π_H.

A priori, the above axioms **(Dis)**, **(Bij)**-**(Bij')**,**(St)**, and **(ECI)**-**(ECI')** are not enough to uniquely specify a supercuspidal local Langlands correspondence Π for G^* even under the specification of the set of Π-accessible representations. The goal of our main theorem is to explain a sufficient extra condition which does uniquely specify a supercuspidal local Langlands correspondence.

In the statement of this condition we need to assume an extra property of G. Namely, we say that G^* is *good* if for every elliptic hyperendoscopic group H of G^* we have:

(Mu) There exists a set S^H of dominant cocharacters of $H_{\overline{F}}$ with the following property. Let ψ_1^H, ψ_2^H be any pair of supercuspidal parameters of H such that for all dominant cocharacters $\mu \in S^H$, we have an equivalence $r_{-\mu} \circ \psi_1^H \sim r_{-\mu} \circ \psi_2^H$. Then $\psi_1^H \sim \psi_2^H$.

Here $r_{-\mu}$ is the representation of ${}^L H$ as defined in [Kot84a, (2.1.1)].We say that G is *good* if G^* is. We call a set S^H as in assumption **(Mu)** *sufficient*.

To this end, let us define a *Scholze–Shin datum* $\{f^\mu_{\tau,h}\}$ for G to consist of the following data for each elliptic hyperendoscopic group H of G:

- A compact open subgroup $K^H \subset H(F)$,
- A sufficient set S^H of dominant cocharacters of $H_{\overline{F}}$,
- For each $\mu \in S^H$ of with reflex field E_μ, each $\tau \in W_{E_\mu}$, and each $h \in \mathscr{H}(K^H)$, a function $f^\mu_{\tau,h} \in \mathscr{H}(H(F))$.

Let us say that a supercuspidal local Langlands correspondence for G satisfies the *Scholze–Shin equations* relative to the Scholze–Shin datum $\{f^\mu_{\tau,h}\}$ if the following holds:

(SS) For all elliptic hyperendoscopic groups H, all $h \in \mathscr{H}(K^H)$, all $\mu \in S^H$, and all parameters ψ^H of H one has that

$$S\Theta_{\psi^H}(f^\mu_{\tau,h}) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi^H)(\chi_\mu))S\Theta_{\psi^H}(h), \tag{16}$$

where $\chi_\mu := |\cdot|^{-\langle \rho,\mu\rangle}$ and ρ is the half-sum of the positive roots of H (for a representation V and character χ we denote by $V(\chi)$ the character twist of V by χ).

Remark 3.2. The conditions **(Mu)** and **(SS)** allow the set S^H to include non-minuscule cocharacters. As we remarked in the introduction, the only known candidates for the functions $f^{\mu}_{\tau,h}$ have been constructed with certain non-trivial assumptions (e.g. see [Sch13a] and[You19]). In particular, they assume that μ is minuscule. That said, it seems reasonable to the authors that one could extend these definitions to work in the case of arbitrary μ using the theory of the moduli of mixed characteristic shtukas.

Similarly, the only known method to show that the aforementioned set of functions $\{f^{\mu}_{\tau,h}\}$ form a Scholze–Shin datum (i.e. that the Scholze–Shin equations hold for these candidate functions) uses global techniques involving Shimura varieties and therefore necessitates the cocharacter μ to be minuscule. It again seems conceivable to the authors that future work on the cohomology of moduli spaces of mixed characteristic shtukas, perhaps in analogy with the work of [GL17], could lead to purely local proofs, freeing us from this constraint.

Because the minuscule condition is not necessary for the arguments in this paper and shows up more as an artifact of arguments using global methods to prove **(SS)**, we have chosen to state our definitions in the above generality.

We then have the following result:

Theorem 3.3. *Let G be a good group and suppose Π^i for $i = 1, 2$ are supercuspidal local Langlands correspondences for (G, z_{iso}) such that*

1. *For every elliptic hyperendoscopic group H of G the set of singly Π^1_H-accessible representations is contained in the set of Π^2_H-accessible representations.*

2. *There exists a Scholze–Shin datum $\{f^{\mu}_{\tau,h}\}$ such that Π^i satisfies* **(SS)** *relative to $\{f^{\mu}_{\tau,h}\}$ for $i = 1, 2$.*

Then $\Pi^1 = \Pi^2$ and for every (H, z), either equal to $(H, 1)$ where H is an elliptic hyperendoscopic group of G or equal to (G, z_{iso}), and choice of Whittaker datum $\mathfrak{w}_H$, the bijections $\iota^i_{\mathfrak{w}_H}$ for $i = 1, 2$ agree.

Remark 3.4. In this paper we have considered only G that arise as extended pure inner twists of G^* (e.g. see [Kal16a]). In general, the map

$$\mathbf{B}(G^*)_{\text{bas}} \to \text{Inn}(G^*), \tag{17}$$

where $\text{Inn}(G^*) := \text{im}[H^1(F, G^*_{\text{ad}}(\overline{F})) \to H^1(F, \text{Aut}(G^*)(\overline{F})]$ denotes the set of inner twists of G^*, need not be surjective. However, when G^* has connected center, this map will be surjective (see [Kal16a, pg.20]). In general, one can likely consider all inner twists by adapting the arguments of this paper to the language of rigid inner twists as in [Kal16b] (cf. [Kal16a]).

4 Atomic stability of L-packets

Before we begin the proof of Theorem 3.3 in earnest, we first discuss the following extra assumption one might make on a supercuspidal local Langlands correspondence Π for the group G which, for this section, we assume is quasi-split. Namely, let us say that Π possesses *atomic stability* if the following condition holds:

(AS) If $S = \{\pi_1, ..., \pi_k\}$ is a finite subset of Π-accessible elements of $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ and $\{a_1, ..., a_k\}$ is a set of complex numbers such that $\Theta := \sum_{i=1}^{k} a_i \pi_i$ is a stable distribution, then there is a partition

$$S = \Pi_{\psi_1}(G) \sqcup \cdots \sqcup \Pi_{\psi_n}(G) \tag{18}$$

such that

$$\Theta = \sum_{j=1}^{n} b_j S\Theta_{\psi_j} \tag{19}$$

(i.e. that a_i is constant on $\Pi_{\psi_i}(G)$).

We then have the following result:

Proposition 4.1. *Let Π be supercuspidal local Langlands correspondence for a group G. Then, Π automatically possesses atomic stability.*

Proposition 4.1 will follow from the following a priori weaker proposition. To state it we make the following definitions. For supercuspidal L-parameters $\psi_1, \ldots, \psi_n$ we denote by $D(\psi_1, \ldots, \psi_n)$ the $\mathbb{C}$-span of the distributions Θ_π for $\pi \in \Pi_G(\psi_1) \cup \cdots \cup \Pi_G(\psi_n)$ and let $S(\psi_1, \ldots, \psi_n)$ be the subspace of stable distributions in $D(\psi_1, \ldots, \psi_n)$.

Proposition 4.2. *For any finite set of supercuspidal L-parameters $\{\psi_1, \ldots, \psi_n\}$ one has that $\{S\Theta_{\psi_1}, \ldots, S\Theta_{\psi_n}\}$ is a basis for $S(\psi_1, \ldots, \psi_n)$.*

Let us note that this proposition actually implies Proposition 4.1. Indeed, since each $\pi_i \in S$ is accessible we know that we can enlarge S to be a union $\Pi_{\psi_1}(G) \sqcup \cdots \sqcup \Pi_{\psi_n}(G)$ of L-packets. Proposition 4.1 is then clear since every stable distribution in the span of S is contained in $S(\psi_1, \ldots, \psi_n)$.

Before we proceed with the proof of Proposition 4.2 we establish some further notation and basic observations. For an π element of $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ we denote by f_π the locally constant $\mathbb{C}$-valued function on $G(F)^{\mathrm{reg}}$ given by the Harish-Chandra regularity theorem. We then obtain a linear map

$$R : D(\mathrm{Irr}^{\mathrm{sc}}(G(F))) \to C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \tag{20}$$

given by linearly extending the association $\Theta_\pi \mapsto f_\pi |_{G(F)^{\mathrm{ell}}}$. Here $D(\mathrm{Irr}^{\mathrm{sc}}(G(F)))$ is the $\mathbb{C}$-span of the distributions on $\mathcal{H}(G(F))$ of the form Θ_π for $\pi \in \mathrm{Irr}^{\mathrm{sc}}(G(F))$. We also have averaging maps

$$\mathrm{Avg} : C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \to C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \tag{21}$$

given by

$$\mathrm{Avg}(f)(\gamma) := \frac{1}{n_\gamma} \sum_{\gamma'} f(\gamma') \tag{22}$$

where γ' runs over representatives of the conjugacy classes of $G(F)$ stably equal to the conjugacy class of γ and n_γ is the number of such classes (which is finite since F is a p-adic field).

We then have the following well-known lemma concerning R:

Lemma 4.3 ([Kaz86, Theorem C])**.** *The linear map R is injective.*

In addition, we have the following observation concerning the interaction between R and Avg, which follows from the well-known fact that Θ is stable implies that $R(\Theta)$ is stable:

Lemma 4.4. *Let $\Theta \in D(\mathrm{Irr}^{\mathrm{sc}}(G(F)))$ be stable as a distribution. Then,* $\mathrm{Avg}(R(\Theta)) = R(\Theta)$.

We may now proceed to the proof of Proposition 4.2:

Proof. (Proposition 4.2) By assumption **(Bij)**, the set of virtual characters $S\Theta^s_{\psi_i}$, as s runs through representatives for the conjugacy classes in $\overline{C_\psi}$ and i runs through $\{1, \ldots, n\}$, is a basis of $D(\psi_1, \ldots, \psi_n)$. It suffices to show this in the case when $n = 1$ in which case it is clear. Indeed, writing just ψ instead of ψ_1, we see that it suffices to note that the matrix $(\langle \pi, s \rangle)$, where π runs through the elements of $\Pi_\psi(G)$, is unitary, and thus invertible, by the orthogonality of characters.

We next show that for any supercuspidal L-parameter ψ and any non-trivial s in $\overline{C_\psi}$ we have that $\mathrm{Avg}(R(S\Theta^s_\psi)) = 0$. Indeed, we begin by observing that by [HS12, Lemma 6.20] we have that

$$\mathrm{Avg}(R(S\Theta^s_\psi))(\gamma) = \frac{1}{n_\gamma} \sum_{\gamma'} \sum_{\gamma_H \in X(\gamma')/\sim_{\mathrm{st}}} \Delta(\gamma_H, \gamma') \left| \frac{D_H(\gamma_H)}{D_G(\gamma')} \right| S\Theta_{\phi^H}(\gamma_H) \tag{23}$$

where here γ' travels over the set of conjugacy classes of $G(F)$ stably equal to the conjugacy class of γ and, as in loc. cit., $X(\gamma')$ is the set of conjugacy classes in $H(F)$ that transfer to γ, and $\Delta(\gamma_H, \gamma')$ is the usual transfer factor, and D denotes the discriminant function.

Let us note that we can rewrite this sum as

$$\frac{1}{n_\gamma} \sum_{\gamma_H \in X(\gamma)/\sim_{\mathrm{st}}} \left(\sum_{\gamma'} \Delta(\gamma_H, \gamma') \left| \frac{D_H(\gamma_H)}{D_G(\gamma')} \right| \right) S\Theta_{\phi^H}(\gamma_H). \tag{24}$$

because $X(\gamma')/\sim_{\mathrm{st}}$ is independent of the choice of γ'.

Note that $D_G(\gamma') = D_G(\gamma)$ for all γ' stably conjugate to γ (since $D_G(\gamma')$ is defined in terms of the characteristic polynomial of $\mathrm{Ad}(\gamma')$) and thus we can further rewrite this as

$$\frac{1}{n_\gamma} \sum_{\gamma_H \in X(\gamma)/\sim_{\mathrm{st}}} \left| \frac{D_H(\gamma_H)}{D_G(\gamma)} \right| \left(\sum_{\gamma'} \Delta(\gamma_H, \gamma') \right) S\Theta_{\phi^H}(\gamma_H) \tag{25}$$

and so it suffices to show that this inner sum $\sum_{\gamma'} \Delta(\gamma_H, \gamma')$ is zero.

For $\gamma' \sim_{st} \gamma$, we have

$$\Delta(\gamma_H, \gamma') = \langle \mathrm{inv}(\gamma, \gamma'), s \rangle \Delta(\gamma_H, \gamma), \tag{26}$$

where $\mathrm{inv}(\gamma, \gamma') \in \mathfrak{K}(I_\gamma/F)^D$ (as in [Shi10, §2.2]). Since γ is elliptic, $\gamma' \mapsto \mathrm{inv}(\gamma, \gamma')$ gives a bijection between F-conjugacy classes in the stable conjugacy class of γ and $\mathfrak{K}(I_\gamma/F)^D$. Hence

$$\sum_{\gamma'} \Delta(\gamma_H, \gamma') = \Delta(\gamma_H, \gamma) \sum_{\chi \in \mathfrak{K}(I_\gamma)^D} \chi(s). \tag{27}$$

In particular, it suffices to show that s gives a nontrivial element of $\mathfrak{K}(I_\gamma/F)$. Since (H, s, η) is a nontrivial elliptic endoscopic datum and γ is elliptic, this follows from [Shi10, Lemma 2.8].

Now, since the set $\{S\Theta_{\psi_1}, \ldots, S\Theta_{\psi_n}\}$ is independent (by assumption **(Dis)**) it suffices to show that this set generates $S(\psi_1, \ldots, \psi_n)$. But, this is now clear since if $\Theta \in S(\psi_1, \ldots, \psi_n)$ then we know by Lemma 4.4 that $\mathrm{Avg}(R(\Theta)) = R(\Theta)$. On the other hand, writing

$$\Theta = \sum_{i=1}^{n} \sum_{s} a_{is} S\Theta^{s}_{\psi_i} \tag{28}$$

we see from the above discussion, as well as combining assumption **(St)** with Lemma 4.4, that

$$\mathrm{Avg}(R(\Theta)) = \sum_{i=1}^{n} R(Sa_{ie}\Theta_{\psi_i}) = R\left(\sum_{i=1}^{n} a_{ie} S\Theta_{\psi_i} \right) \tag{29}$$

(identifying $S\Theta_{\psi_i}$ with $S\Theta^{e}_{\psi_i}$ where e is the identity conjugacy class in $\overline{C_\psi}$). The claim then follows from Lemma 4.3. □

5 Proof of main result

Let us begin by explaining that it suffices to assume G is quasi-split. Indeed, note that the assumptions of Theorem 3.3 are also satisfied for (G, z_{iso}) equal to $(G^*, 1)$ and so, in particular, if we have proven the theorem in the case of $(G^*, 1)$ then we know that $\Pi^1_{G^*} = \Pi^2_{G^*}$. Now, let ψ be any supercuspidal L-parameter for G. By assumption **(ECI')** we have that

$$S\Theta^1_\psi(h) = S\Theta^1_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_\psi(h) \tag{30}$$

for all $h \in \mathscr{H}(G(F))$ and where the superscripts correspond to those of Π^i. By independence of characters, this implies that $\Pi^1_{(G,z_{\mathrm{iso}})}(\psi) = \Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$. It remains to show that $\iota^1_{\mathfrak{w}_H} = \iota^2_{\mathfrak{w}_H}$. Since each $\iota^i_{\mathfrak{w}_G}(\pi)$ is algebraic, it suffices to show that for all $\pi \in \Pi^1_{(G,z_{\mathrm{iso}})}(\psi) = \Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$ one has that $\langle \pi, s\rangle^1_{\mathfrak{w}_H} = \langle \pi, s\rangle^2_{\mathfrak{w}_H}$ for all $s \in C_\psi$. By independence of characters, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all $s \in C_\psi$. By the standard bijection $(H, s, {}^L\eta, \psi^H) \Longleftrightarrow (\psi, s)$ (cf. [BM20, Prop. 2.10]) and the **(Ext)** assumption, each such s comes from an extended elliptic endoscopic datum $(H, s, {}^L\eta)$. Hence by **(ECI')** we have reduced to the quasi-split setting. We now work in the situation when $(G, z_{\mathrm{iso}}) = (G^*, 1)$.

Let us begin with the following lemma:

Lemma 5.1. *Suppose that H is an elliptic hyperendoscopic group of G and suppose that $\Pi^1_H(\psi)$ is a singleton set $\{\pi\}$. Then, in fact, $\{\pi\} = \Pi^2_H(\psi)$.*

Proof. Since $\{\pi\}$ is a supercuspidal packet for Π^1_H, we have by assumption **(St)** that Θ_π is stable. By the assumption of the theorem, π is Π^2_H-accessible and since Π^2_H satisfies **(AS)** (by the contents of §4), we have $\{\pi\} = \Pi^2_H(\psi')$ for some supercuspidal L-parameter ψ' of H. Then, by the assumption of the theorem we have that

$$\mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi)(\chi_\mu))\,\mathrm{tr}(h \mid \pi) = \mathrm{tr}(f^\mu_{\tau,h} \mid \pi) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi')(\chi_\mu))\,\mathrm{tr}(h \mid \pi) \tag{31}$$

In particular, choosing $h \in \mathscr{H}(K^H)$ such that $\mathrm{tr}(h \mid \pi) \neq 0$ and letting τ vary we deduce that

$$\mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi)(\chi_\mu)) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi')(\chi_\mu)) \tag{32}$$

for all $\tau \in W_E$. This implies, since ψ is supercuspidal so that $r_{-\mu} \circ \psi$ and $r_{-\mu} \circ \psi'$ are semi-simple, that $r_{-\mu} \circ \psi \sim r_{-\mu} \circ \psi'$ for all $\mu \in S^H$. By our assumption that S^H is sufficient, we deduce that $\psi \sim \psi'$. In particular, $\{\pi\} = \Pi^2_\psi(H)$ as desired. □

Lemma 5.2. *Let H be an elliptic hyperendoscopic group for G. Let ψ be a supercuspidal parameter for H and suppose $\overline{C_\psi} \neq \{1\}$. If ρ is an irreducible representation of $\overline{C_{\psi^H}}$ then there exists a nontrivial $\overline{s} \in \overline{C_\psi}$ such that the trace character χ_ρ of ρ satisfies $\mathrm{tr}(\overline{s} \mid \rho) \neq 0$.*

Proof. Suppose ρ vanishes on all nontrivial $\overline{s}$. Then we have

$$1 = \langle \chi_\rho, \chi_\rho\rangle = \frac{1}{|\overline{C_\psi}|} \sum_{\overline{s} \in \overline{C_\psi}} \chi_\rho(\overline{s})^2 = \frac{1}{|\overline{C_\psi}|}\chi_\rho(1)^2 = \frac{1}{|\overline{C_\psi}|}\dim(\rho)^2, \tag{33}$$

so that $|\overline{C_\psi}| = \dim(\rho)^2$. But every irreducible representation ρ' of $\overline{C_\psi}$ is isomorphic to an irreducible factor appearing with multiplicity $\dim(\rho')$ in the regular representation of $\overline{C_\psi}$, which has dimension $|\overline{C_\psi}|$. Hence ρ must be the unique irreducible representation of $\overline{C_\psi}$, which implies that ρ is isomorphic to the trivial representation, and hence that $|\overline{C_\psi}| = 1$ contrary to assumption. □

We now explain the proof of Theorem 3.3 in general:

Proof. (of Theorem 3.3) We prove this by inducting on the number of roots k for elliptic hyperendoscopic groups H of G. If $k = 0$ then H is a torus. Since every distribution on H is stable, one deduces from assumption **(Dis)** and assumption **(St)** that $\Pi^1_H(\psi)$ is a singleton and thus we are done by Lemma 5.1. Suppose now that the result is true for elliptic hyperendoscopic groups of G with at most k roots. Let H be an elliptic hyperendoscopic group of G with $k+1$ roots and let ψ be a supercuspidal parameter of H. We wish to show that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$. If $\Pi^1_H(\psi)$ is a singleton, then we are done again by Lemma 5.1. Otherwise, we show that $\Pi^1_H(\psi) \subset \Pi^2_H(\psi)$, which by **(Bij)** will imply that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$. By Lemma 5.2, we can find a non-trivial $\overline{s} \in \overline{C_\psi}$ and a lift $s \in C_\psi$ such that $\langle \pi, s\rangle \neq 0$. By definition of $\overline{C_\psi}$, we have that $s \notin Z(\widehat{G})$. Now, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ since then by independence of characters, we deduce that $\pi \in \Pi^2_H(\psi)$ as desired.

To show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all non-trivial $s \in \overline{C_\psi}$ we proceed as follows. We obtain, by combining our assumption **(Ext)** and [BMY19, Proposition I.2.15]) from (ψ, s), an extended elliptic endoscopic quadruple $(H', s, {}^L\eta, \psi^{H'})$ with $\psi^{H'}$ supercuspidal so that $\psi = {}^L\eta \circ \psi^{H'}$. One then has from Assumption **(ECI)** that

$$\Theta^{1,s}_\psi = \Theta^{2,s}_\psi \iff S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}} \tag{34}$$

Moreover, since s is non-central, we know that H' has a smaller number of roots than H and thus $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$ by induction. The conclusion that $\Pi^1 = \Pi^2$ follows.

Let us now show that for any supercuspidal L-parameter ψ one has that $\iota^1_{\mathfrak{w}_H} = \iota^2_{\mathfrak{w}_H}$ for all elliptic hyperendoscopic groups H of G and Whittaker data $\mathfrak{w}_H$ of H. It suffices to show that $\langle \pi, s\rangle^1_{\mathfrak{w}_H} = \langle \pi, s\rangle^2_{\mathfrak{w}_H}$ for all $\pi \in \Pi^1_\psi(H) = \Pi^2_\psi(H)$. By independence of characters, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all $s \in \overline{C_\psi}$. Since $s \in \overline{C_\psi}$, there exists, associated to the pair (ψ, s), a quadruple $(H', s, {}^L\eta, \psi^{H'})$ as in [BMY19, Proposition I.2.15] (again using also assumption **(Ext)**) where H' is an elliptic endoscopic group of H and $\psi^{H'}$ is a parameter such that $\psi = {}^L\eta \circ \psi^{H'}$. By assumption **(ECI)** it suffices to show that $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$, but this follows from the previous part of the argument since we know that $\Pi^1_{H'}(\psi^{H'}) = \Pi^2_{H'}(\psi^{H'})$. The theorem follows. □

6 Examples

In this last section we discuss some examples of where the conditions necessary to apply Theorem 3.3 are satisfied.

6.1 The characterization in the unitary case

We start by discussing the case of unitary groups which was mentioned several times in the introduction. Namely, let F be an extension of $\mathbb{Q}_p$ and let E be a quadratic extension of F. Let us set $U^*_{E/F,n}$ to be the Zariski closure of the set

$$U^*_{E/F,n}(F) = \left\{g \in \mathrm{GL}_{n,E}(E) : gJ_n\sigma(g)^{-1} = J_n\right\}$$

where σ is the unique non-trivial element of $\mathrm{Gal}(E/F)$ and J_n is the anti-diagonal matrix with $a_{i,n-i} = (-1)^{i-1}$. We call this the *quasi-split unitary group* of dimension n associated to the extension E/F.

We call a group G over F a *unitary group* of dimension n associated to the extension E/F if one has $G^* = U^*_{E/F,n}$. If n is odd then any unitary group is automatically quasi-split, but if n is even there is precisely one isomorphism class of unitary group which is not quasi-split. Note though that every inner form of $U^*_{E/F,n}$ can be upgraded to a pure inner twist, and we leave such choice implicit. We say that G is *unramified* if the extension $E/\mathbb{Q}_p$ is unramified.

Let us note that if G is a unitary group of dimension n associated to an extension E/F then it is automatically good. Indeed, note that every elliptic endoscopic group of G is of the form $U^*_{E/F,a} \times U^*_{E/F,b}$ for some $a, b \in \mathbb{N}$ such that $a + b = n$ (e.g. see [Rog90, Proposition 4.6.1]). From this we deduce that the elliptic hyperendoscopic groups of G are given by $U^*_{E/F,P}$ where $P = (n_1, \ldots, n_k)$ is a partition of n and

$$U^*_{E/F,P} := U^*_{E/F,n_1} \times \cdots \times U^*_{E/F,n_m}$$

We note then that G is clearly good as we can take $S^{U^*_{E/F,P}}$ to be $\{\mu_{P,\mathrm{std}}\}$ where

$$\mu_{P,\mathrm{std}} = \mu_{n_1,\mathrm{std}} \times \cdots \times \mu_{n_m,\mathrm{std}}$$

where $\mu_{n_i,\mathrm{std}}$ is the cocharacter corresponding to the standard representation of $U^*_{E/F,n_i}$.

Thus, we deduce the following:

Theorem 6.1. *Let G be a unitary group. Then, for any Scholze–Shin datum $\{f^{\mu_{P,std}}_{\tau,h}\}$ there exists at most one supercuspidal local Langlands correspondence Π for G with a specified set of singly Π_H-accessible representations for all H.*

In particular, let us define a Scholze–Shin datum $\{f^{\mu_P,\mathrm{std}}_{\tau,h}\}$ as in [You19]. Let us then set Π^{Mok} to be the supercuspidal local Langlands correspondence associated to an unramified unitary group as in [Mok15]. Then, the results of [BMY19] and Theorem 3.3 show the following:

Theorem 6.2. *Let G be an unramified unitary group. Then, Π^{Mok} is characterized by the Scholze–Shin datum $\{f^{\mu_P,\mathrm{std}}_{\tau,h}\}$ and the set of singly Π^{Mok}-accessible representations.*

6.2 The characterization in the odd orthogonal case

We now discuss the case of odd special orthogonal groups. Let F be a finite extension of $\mathbb{Q}_p$ and let $n \geqslant 1$ be an integer. By an *odd special orthogonal group* we mean a group G over F of the form $\mathrm{SO}(V, q)$ where (V, q) is a quadratic space over F of odd dimension. We denote by $\mathrm{SO}_{2n+1,F}$ the special odd orthogonal group of the split quadratic space of dimension $2n+1$ which is a split group. Moreover, for each n there is precisely one non-split inner form of $\mathrm{SO}_{2n+1,F}$, and every odd special orthogonal group is an inner form of some $\mathrm{SO}_{2n+1,F}$. In particular, for any special odd orthogonal group G one has that $G^* = \mathrm{SO}_{2n+1,F}$ for some n. We call an odd special orthogonal group *unramified* if F is unramified.

Let us denote by μ_n the unique non-trivial minuscule cocharacter of $\mathrm{SO}_{2n+1,F}$. We then have the following result:

We observe that every special odd orthogonal group is good:

Proposition 6.3. *Let G be an odd special orthogonal group. Then, G is good.*

Proof. Since G being good depends only on G^*, and $G^* = \mathrm{SO}_{2n+1,F}$ for some n we may assume that $G = \mathrm{SO}_{2n+1,F}$. By [GGP12, Theorem 8.1], we can recover ψ from $r \circ \psi$ where ψ any admissible homomorphism $\psi : W_F \to \mathrm{Sp}(2n)(\mathbb{C})$ and r is the standard representation. Let us then note that G satisfies **(Mu)** relative to $\{\mu\}$ since $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ and $r_{-\mu} = r$. However, to prove that G is good we must show that H satisfies **(Mu)** for every elliptic hyperendoscopic group H of G. But, since every such elliptic hyperendoscopic group of G is a product of odd special orthogonal groups (e.g. by [Wal10, §1.8]) we are done. □

As noted in the above proof, every elliptic hyperendoscopic H group of G, an odd special orthogonal group, is a product of odd special orthogonal groups. Let us write

$$H = \mathrm{SO}_{2n_1+1,F} \times \cdots \times \mathrm{SO}_{2n_k+1,F}$$

denote by μ_H the cocharacter

$$\mu_{n_1} \times \cdots \times \mu_{n_k}$$

of H.

Then, we obtain the following:

Theorem 6.4. *Let G be an odd special orthogonal group. Then, for any Scholze–Shin datum $\{f^{\mu_H}_{\tau,h}\}$ there exists at most one supercuspidal local Langlands correspondence Π for G with a specified set of singly Π_H-accessible representations for all H.*

We end by remarking to what extent one might hope that the result Theorem 6.2 extends to the case of odd special orthogonal groups. We begin by noting that a construction of the local Langlands correspondence for odd special orthogonal groups is complete by [Art13]. Moreover, the works of Arthur and [Tai19] prove the global multiplicity formula results. Such multiplicity results play a pivotal role in the proof in [BMY19] that Mok's Langlands correspondence for unramified unitary groups satisfies the Scholze–Shin equations for the data $\{f^{\mu_{P,\mathrm{std}}}_{\tau,h}\}$ from the last section.

Moreover, there are well-studied Shimura data associated to the odd special orthogonal groups over number fields (see [Zhu18]). The cocharacter associated to this Shimura datum is μ_n. In [You19] there are constructed functions $f^{\mu_H}_{\tau,h}$ which serve as candidate Scholze–Shin data. Combining this geometric input with the aforementioned results of Arthur and Taïbi it then seems conceivable to prove that Arthur's local Langlands correspondence for unramified odd special orthogonal group satisfies the Scholze–Shin equations relative to the functions in [You19]. This would then allow one to prove the analogue of Theorem 6.2 for unramified odd special orthogonal groups.

References

[Art13] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013, Orthogonal and symplectic groups. MR 3135650

[BM20] Alexander Bertoloni Meli, *An averaging formula for rapoport-zink spaces*, 2020.

[BMY19] Alexander Bertoloni Meli and Alex Youcis, *The scholze-shin conjecture for unramified unitary groups*, preprint on webpage at https://math.berkeley.edu/~abm/resources/Scholze_Shin_Conjecture_for_Unitary_Groups__No_Endoscopy.pdf, 2019.

[Bor79] Armand Borel, *Automorphic l-functions*, Proc. Symp. Pure Math, vol. 33, 1979, pp. 27–61.

[GGP12] Wee Teck Gan, Benedict H Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical l-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), no. 1, 1–109.

[GL17] Alain Genestier and Vincent Lafforgue, *Chtoucas restreints pour les groupes r\'eductifs et param\'etrisation de langlands locale*, arXiv preprint arXiv:1709.00978 (2017).

[Hen00] Guy Henniart, *Une preuve simple des conjectures de langlands pour gl (n) sur un corps p-adique*, Inventiones mathematicae **139** (2000), no. 2, 439–455.

[HS12] Kaoru Hiraga and Hiroshi Saito, *On l-packets for inner forms of sl_n*, American Mathematical Soc., 2012.

[Kal16a] Tasho Kaletha, *The local Langlands conjectures for non-quasi-split groups*, Families of automorphic forms and the trace formula, Simons Symp., Springer, [Cham], 2016, pp. 217–257. MR 3675168

[Kal16b] ———, *Rigid inner forms of real and p-adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632. MR 3548533

[Kal19] Tasho Kaletha, *Supercuspidal l-packets*, 2019.

[Kaz86] David Kazhdan, *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1–36. MR 874042

[Kot84a] Robert E Kottwitz, *Shimura varieties and twisted orbital integrals*, Mathematische Annalen **269** (1984), no. 3, 287–300.

[Kot84b] ———, *Stable trace formula: Cuspidal tempered terms*, Duke Mathematical Journal **51** (1984), no. 3, 611–650.

[Kot97] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. MR 1485921

[KS99] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190. MR 1687096

[Lan79] R. P. Langlands, *Stable conjugacy: definitions and lemmas*, Canadian J. Math. **31** (1979), no. 4, 700–725. MR 540901

[Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, vol. 235, American Mathematical Society, 2015.

[Rog90] Jonathan David Rogawski, *Automorphic representations of unitary groups in three variables*, no. 123, Princeton University Press, 1990.

[Sch13a] Peter Scholze, *The langlands-kottwitz method and deformation spaces of p-divisible groups*, Journal of the American Mathematical Society **26** (2013), no. 1, 227–259.

[Sch13b] ———, *The local Langlands correspondence for* GL_n *over p-adic fields*, Inventiones Mathematicae **192** (2013), no. 3, 663–715.

[Shi10] Sug Woo Shin, *A stable trace formula for igusa varieties*, Journal of the Institute of Mathematics of Jussieu **9** (2010), no. 4, 847–895.

[SS13] Peter Scholze and Sug Woo Shin, *On the cohomology of compact unitary group Shimura varieties at ramified split places*, Journal of the American Mathematical Society **26** (2013), no. 1, 261–294.

[Tai19] Olivier Taibi, *Arthur's multiplicity formula for certain inner forms of special orthogonal and symplectic groups*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 3, 839–871. MR 3908767

[Wal10] J.-L. Waldspurger, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), no. 1-2, 41–82. MR 2672539

[You19] Alex Youcis, *The langlands-kottwitz-scholze method for deformation spaces of abelian type (in preparation).*

[Zhu18] Yihang Zhu, *The stabilization of the frobenius–hecke traces on the intersection cohomology of orthogonal shimura varieties*, 2018.